Moments of Truncated Gaussians

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Given a Gaussian random variable $x$ with mean $\mu$ and variance $\sigma^2$, we show how to compute

$$f(\mu, \sigma^2, \alpha) = \int_{l}^{h} (ax^2 + bx + c) \mathcal{N}(x|\mu, \sigma^2) dx$$

(1)

and its derivatives with respect to $\mu$ and $\sigma^2$, where $\alpha = [a, b, c, l, h]$ with all elements of the set being real-valued scalars. We introduce the notation $E^h_l[x^m|\mu, \sigma^2]$ to indicate the truncated expectation $\int_{l}^{h} x^m \mathcal{N}(x|\mu, \sigma^2) dx$, where $m$ is a non-negative integer. We can then express the moments of a truncated and re-normalized Gaussian distribution.

We now give the derivatives of each truncated moment $E^h_l[x^m|\mu, \sigma^2]$ with respect to $\mu$ and $\sigma^2$. These are all the derivatives needed to complete the definition of the generalized EM algorithm (Algorithm 1).

$$\frac{\partial E^h_l[x^0|\mu, \sigma^2]}{\partial \mu} = \frac{1}{\sigma^2} \left( \phi(\tilde{l}) - \phi(\tilde{h}) \right)$$

(5)

$$\frac{\partial E^h_l[x^0|\mu, \sigma^2]}{\partial \sigma^2} = \frac{1}{2\sigma^2} \left( \tilde{l} \phi(\tilde{l}) - \tilde{h} \phi(\tilde{h}) \right)$$

(6)

$$\frac{\partial E^h_l[x^1|\mu, \sigma^2]}{\partial \mu} = \frac{1}{\sigma} \left( \tilde{l} \phi(\tilde{l}) - \tilde{h} \phi(\tilde{h}) \right) + \Phi(\tilde{h}) - \Phi(\tilde{l})$$

(7)

$$\frac{\partial E^h_l[x^1|\mu, \sigma^2]}{\partial \sigma^2} = \frac{\tilde{l}^2 + \sigma^2 - \mu^2}{2\sigma^3} \phi(\tilde{l}) - \frac{h^2 + \sigma^2 - h^2}{2\sigma^3} \phi(\tilde{h})$$

(8)

$$\frac{\partial E^h_l[x^2|\mu, \sigma^2]}{\partial \mu} = \frac{1}{\sigma} \left( (\tilde{l}^2 + 2\sigma^2) \phi(\tilde{l}) - (h^2 + 2\sigma^2) \phi(\tilde{h}) \right) + 2\mu \left( \Phi(\tilde{h}) - \Phi(\tilde{l}) \right)$$

(9)

$$\frac{\partial E^h_l[x^2|\mu, \sigma^2]}{\partial \sigma^2} = \frac{\tilde{l}^4 + 2\sigma^2 \tilde{l} - \tilde{l}^2 \mu}{2\sigma^3} - \frac{h^4 + 2\sigma^2 \tilde{h} - h^2 \mu}{2\sigma^3} \phi(\tilde{h}) + \Phi(\tilde{h}) - \Phi(\tilde{l})$$

(10)