Fast yet Simple Natural-Gradient Descent for Variational Inference

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The Goal of My Research

“To understand the fundamental principles of learning from data and use them to develop algorithms that can learn like living beings.”

Learning by exploring

at the age of 6 months

Converged
at the age
of
12 months

Transfer Learning at 14 months

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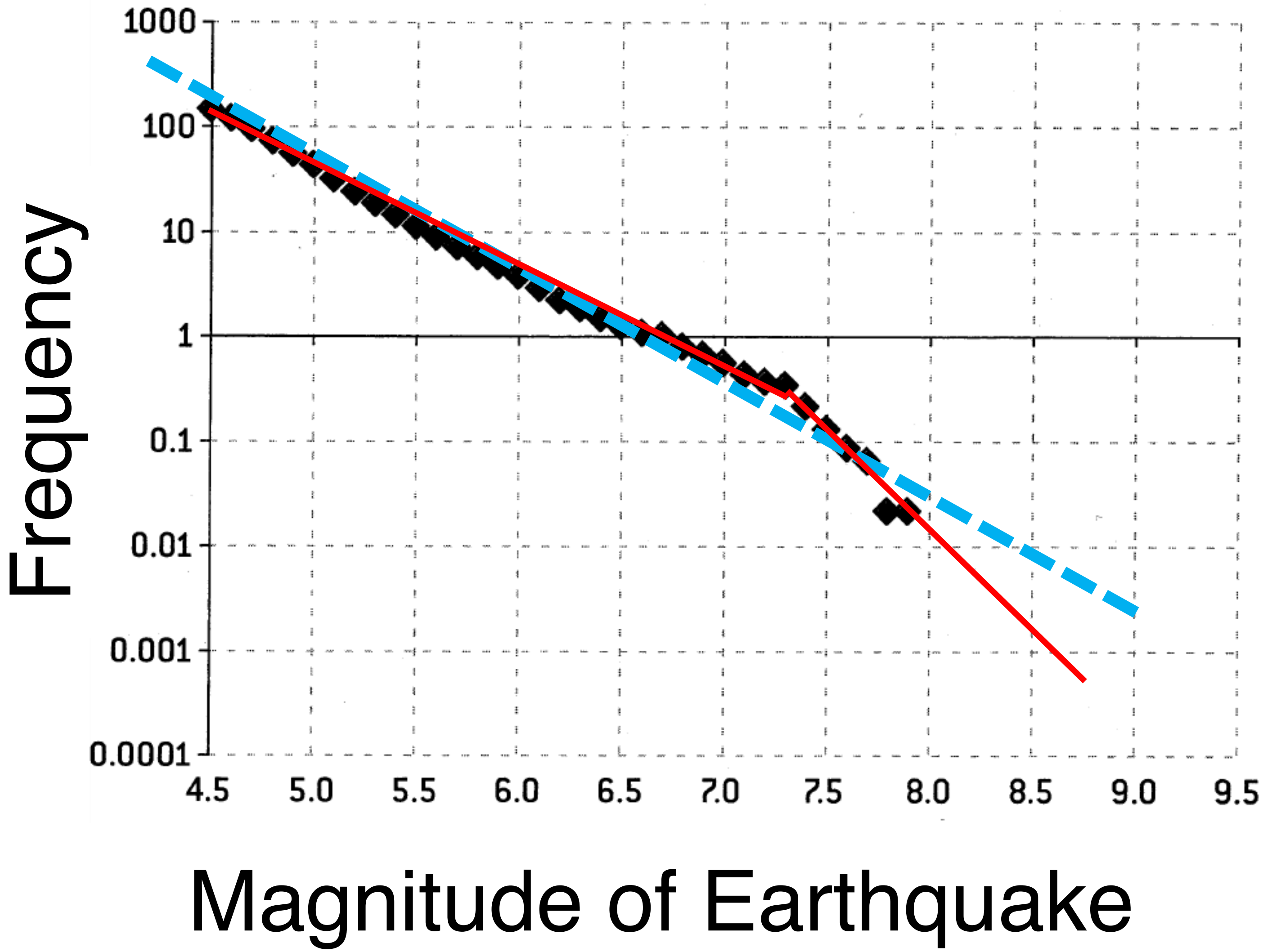
Bayesian Inference

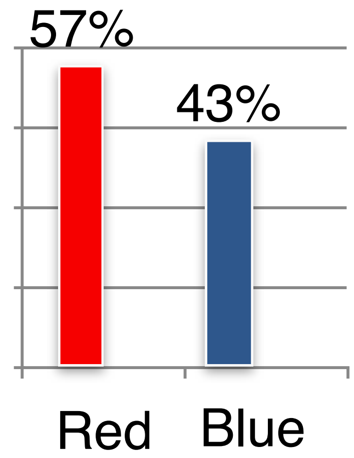
- Compute the posterior distribution
  - Instead of just a point estimate (e.g. MLE).
- A natural representation of all the past information which can then be sequentially updated with new information
  - useful for active learning, sequential experiment design, continual learning, RL.
  - But also for global optimization, causality, etc.
  - Eventually, for ML methods which can learn like humans (data efficient, robust, causal).

Uncertainty in Deep Learning

To estimate the confidence in the predictions of a deep-learning system

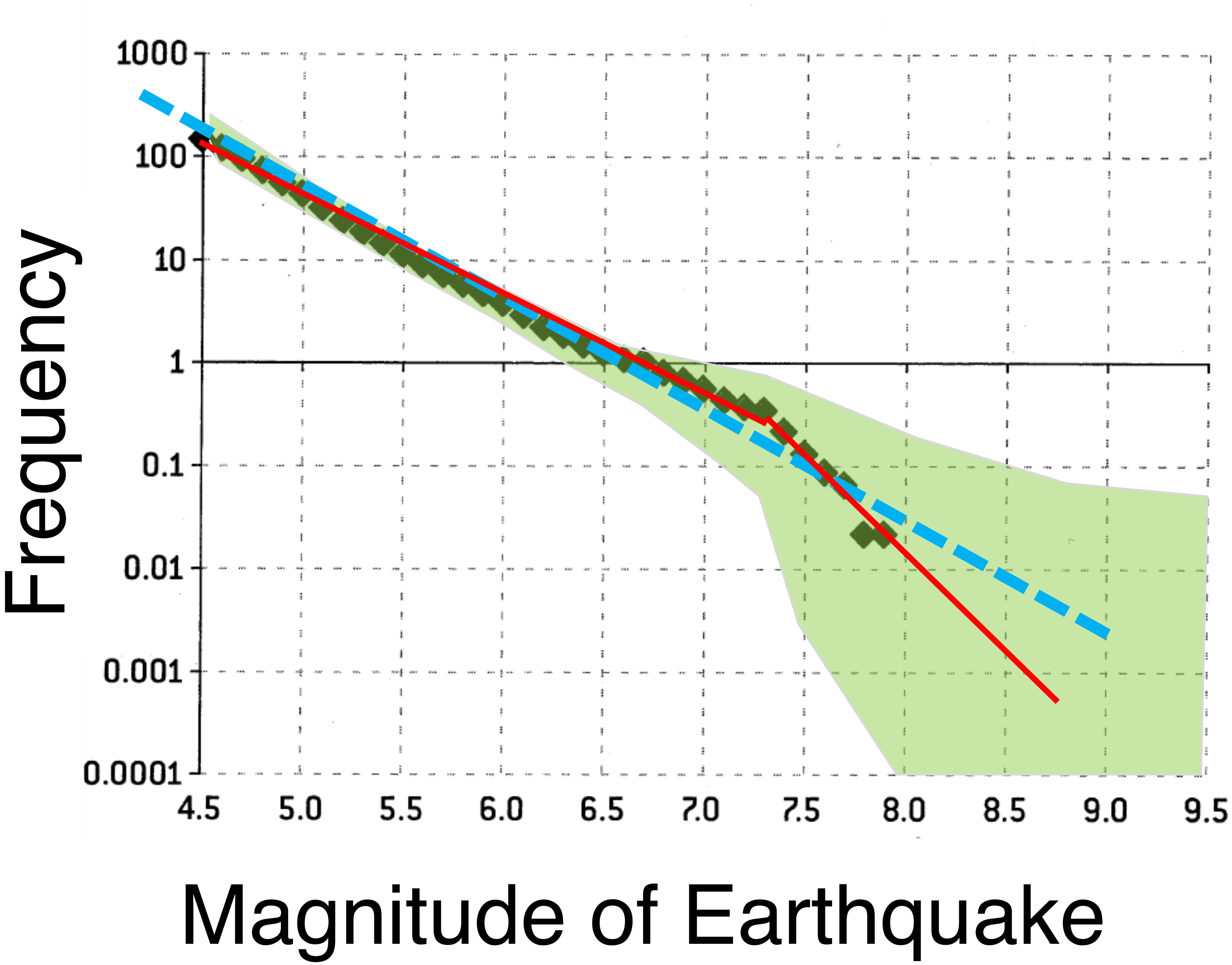
Example: Which is a Better Fit?





Real data from Tohoku (Japan). Example taken from Nate Silver's book "The signal and noise"

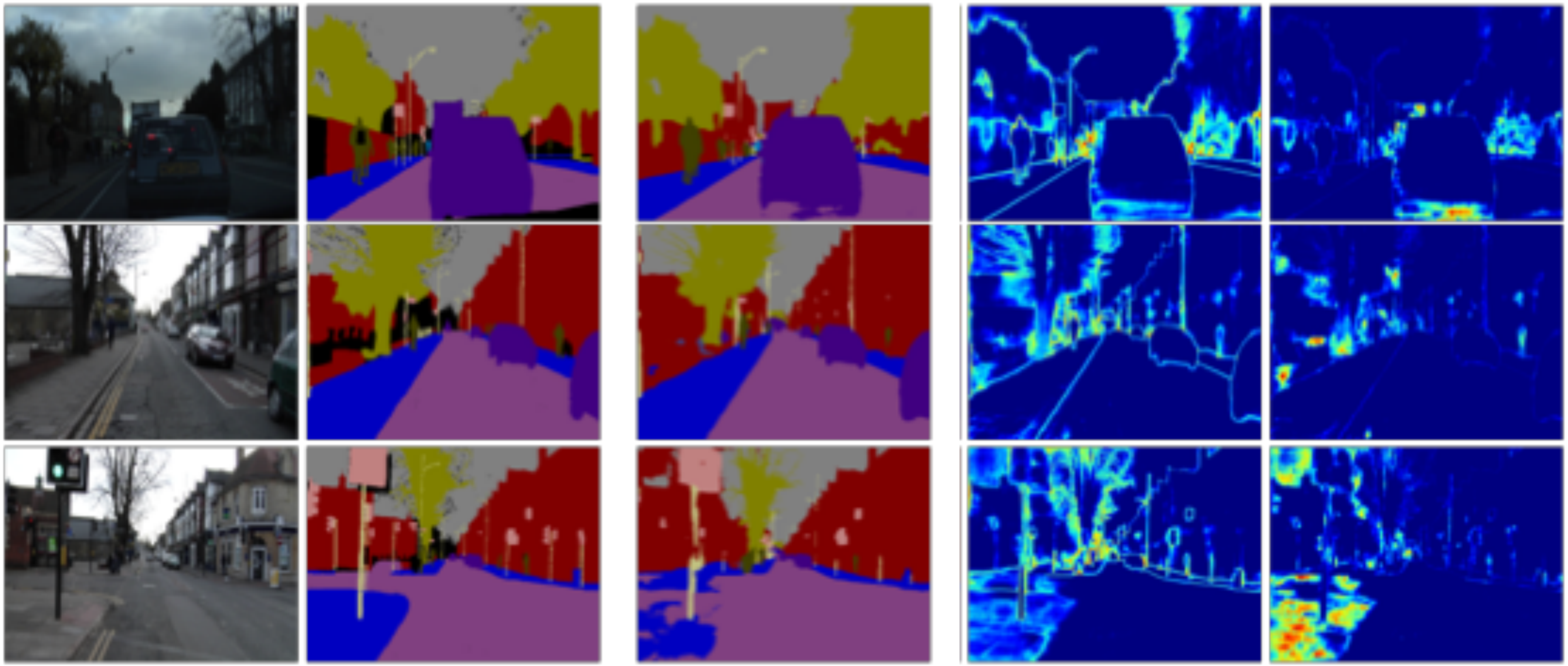
Example: Which is a Better Fit?



When the data is scarce and noisy, e.g., in medicine, and robotics.

Uncertainty for Image Segmentation

Image Truth Prediction Uncertainty



(a) Input Image (b) Ground Truth (c) Semantic Segmentation (d) Aleatoric Uncertainty (e) Epistemic Uncertainty

(taken from Kendall et al. 2017)

Variational Inference (VI)

- Approximate the posterior using optimization
  - Popular in reinforcement learning, unsupervised learning, online learning, active learning etc.
- We need accurate VI algorithms that are
  - general (apply to many models),
  - scalable (for large data and models),
  - fast (converge quickly),
  - simple (easy to implement).
- This talk: New algorithms with such features.

Gradient vs Natural-Gradient

- Gradient Descent (GD)
  - Rely on stochastic and automatic gradients.
  - Simple, general, and scalable, but can have suboptimal convergence.
  - Practical VI (2011), Black-box VI (2014), Bayes by backprop (2015), ADVI (2015), and many more.
- Natural-Gradient Descent (NGD)
  - Fast convergence, but computationally difficult, therefore not simple, general, and scalable
  - (Sato (2001), Riemannian CG (2010), Stochastic VI (2013), etc.
- Fast and simple NGD for complex models, such as those containing deep networks.

Talk Outline

- Variational Inference with gradient descent and natural-gradient descent.
- NGD with Conjugate-Computation VI
  - Generalization of forward-backward algorithms, SVI, Message Passing (AIstats 2017).
  - Deep Nets (ICML 2018, NeurIPS 2018).
- Generalizations and Extensions,
  - Structured VAEs (ICLR 2018), Mixture of Exponential Family approximations, Evolution strategy (ArXiv 2017), etc.

Variational Inference

Gradient Descent (GD)

Vs

Natural-Gradient Descent (NGD)

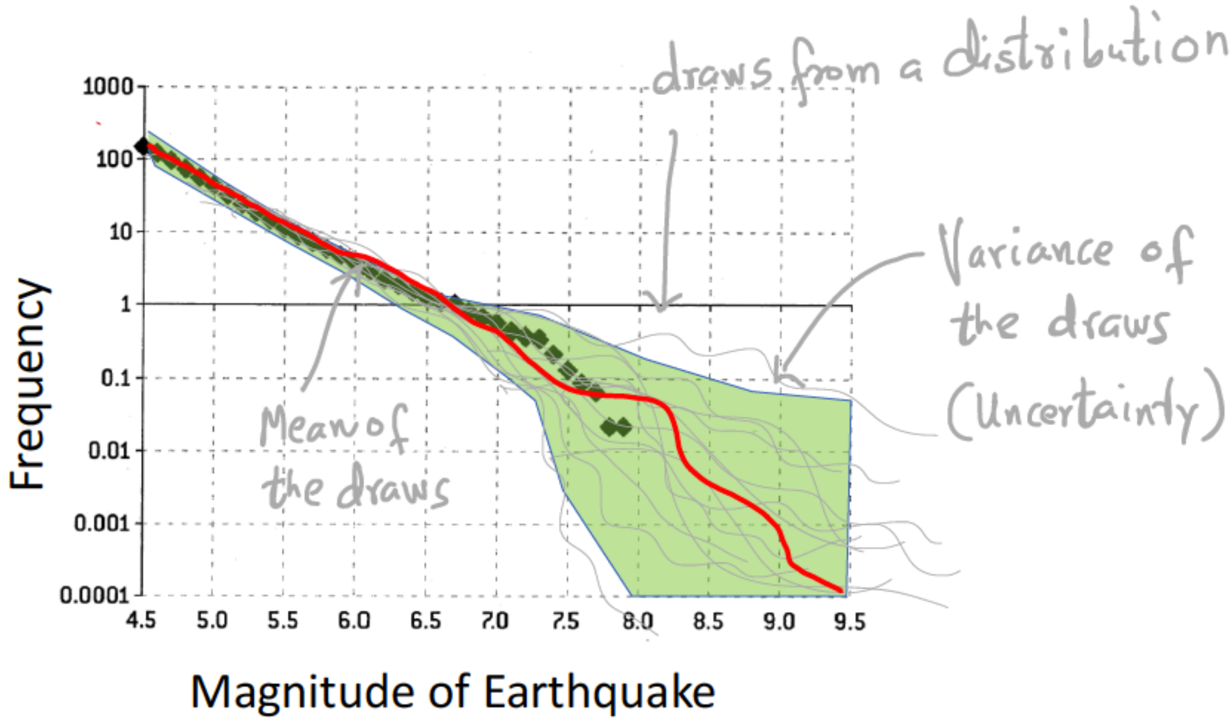
A Naïve Method

Data

$$p(\mathcal{D}|\theta) = \prod_{i=1}^{N} p(y_i|f_\theta(x_i))$$

Output Input

Parameters Neural network



Generate

$$\theta \sim p(\theta)$$

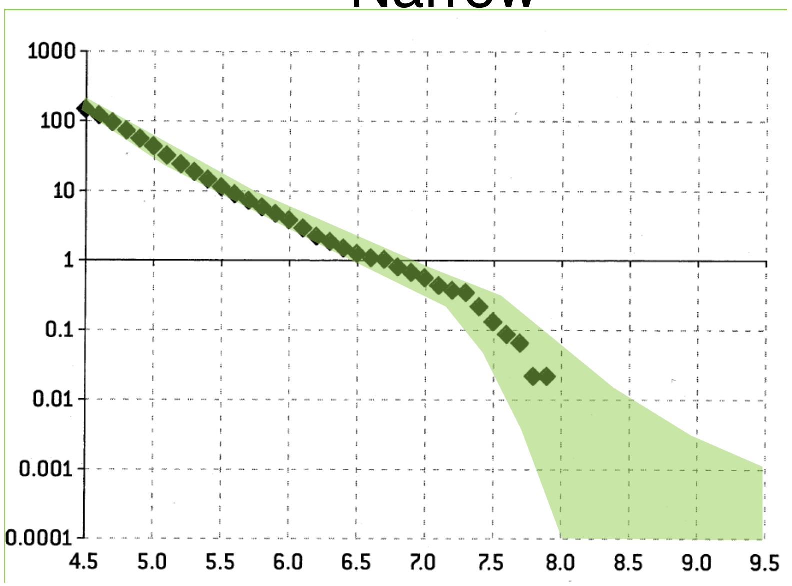
Prior distribution

Bayesian Inference

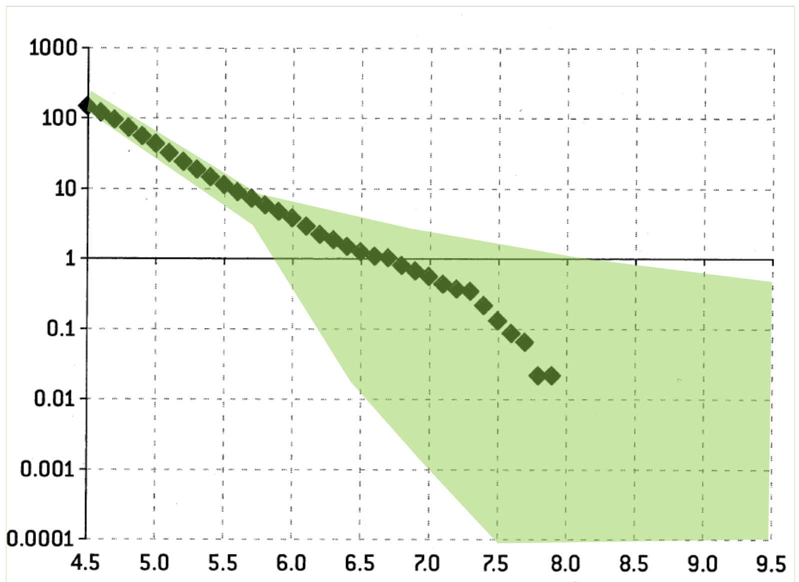
Bayes' rule : $$p(\theta|\mathcal{D}) = \frac{p(\mathcal{D}|\theta)p(\theta)}{\underbrace{\int p(\mathcal{D}|\theta)p(\theta)d\theta}_{\text{Intractable integral}}}$$

Posterior distribution

Narrow



Wide



Variational Inference

Parameters

$$p(\theta|\mathcal{D}) = \frac{p(\mathcal{D}|\theta)p(\theta)}{\underbrace{\int p(\mathcal{D}|\theta)p(\theta)d\theta}_{\text{Variational Approximation}}}$$

Data

Intractable integral

Variational Approximation

$$\approx q_\lambda(\theta) = \mathrm{ExpFamily}(\lambda)$$

Natural parameters

Maximize the Evidence Lower Bound (ELBO):

$$\max_\lambda \mathcal{L}(\lambda) := \mathbb{E}_{q_\lambda}\Big[\log p(\mathcal{D},\theta) - \log q_\lambda(\theta)\Big]$$

Gradient descent (GD) : $\lambda \leftarrow \lambda + \rho \nabla_\lambda \mathcal{L}$

VI with Natural-Gradient Descent

Sato 2001, Honkela et al. 2010, Hoffman et.al. 2013

NGD: $$\lambda \leftarrow \lambda + \rho \underbrace{F(\lambda)^{-1} \nabla_\lambda \mathcal{L}}_{}$$ Natural Gradient

Fisher Information Matrix (FIM)

$$F(\lambda) := \mathbb{E}_{q_\lambda}\left[\nabla \log q_\lambda(\theta) \nabla \log q_\lambda(\theta)^\top\right]$$

- Fast convergence due to optimization in Riemanniam manifold (not Euclidean space).
- But requires additional computations.
- Can we simplify/reduce this computation?

Can we simplify NGD computation? Yes, by using algorithms such as message passing/ backprop.

Conjugate-Computation VI

Khan and Lin, AI-STATS 2017

The key idea: Expectation Parameters

Expectation/moment/
mean parameters

$$\mu := \mathbb{E}_{q_\lambda}[\underbrace{\phi(\theta)}]$$

Sufficient statistics

For Gaussians, it's mean and correlation matrix

$$\mathbb{E}_{q_\lambda}[\theta] = m \qquad \mathbb{E}_{q_\lambda}[\theta\theta^\top] = mm^\top + V$$

A key relationship: $F(\lambda)^{-1}\nabla_\lambda \mathcal{L} = \nabla_\mu \mathcal{L}$

Natural Gradient wrt
natural parameter

Gradient wrt expectation
parameter

NGD : $\lambda \leftarrow \lambda + \rho \nabla_\mu \mathcal{L}$

Conjugate-Computation VI (CVI)

$$\lambda \leftarrow \lambda + \rho \nabla_{\mu} \mathcal{L}$$

- In a "conjugate" model, this is equivalent to simply adding the natural parameters of the factors of a model.
- This is a type of conjugate computation, and enables "simple" updates for complex models.

CVI on Bayesian Linear Regression

$$q_\lambda(\theta) := \mathcal{N}(m, V)$$

$$\mathbb{E}_q\left[\underbrace{\overset{\text{likelihood}}{(y - X\theta)^\top (y - X\theta)}}_{-\mathbb{E}_{q_\lambda}[\theta]^\top X^\top y + \mathrm{trace}\left[X^\top X \mathbb{E}_{q_\lambda}[\theta\theta^\top]\right]} + \overset{\text{prior}}{\gamma\theta^\top\theta} \overset{\text{approx}}{- \log q_\lambda(\theta)}\right]$$

$$\begin{matrix} \nabla_{\mathbb{E}_{q_\lambda}[\theta]} = \\ \nabla_{\mathbb{E}_{q_\lambda}[\theta\theta^\top]} = \end{matrix} \begin{pmatrix} -X^\top y & + & 0 & - & V^{-1}m \\ X^\top X & + & \gamma I & - & V^{-1} \end{pmatrix}$$

Expectation params Natural Gradient

NGD == Newton's Method

$$m \leftarrow (1-\rho)m - \rho \underbrace{\left[X^\top X + \gamma I\right]^{-1} X^\top y}_{\text{Least-square solution}}$$

For rho=1, converges in 1 step (Newton's method).

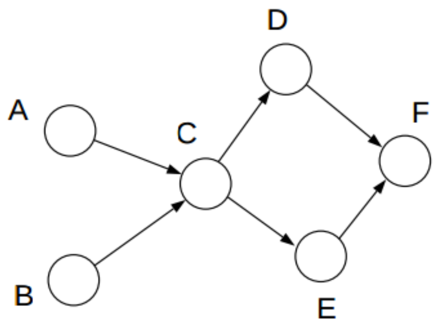
Gradient descent is suboptimal:

$$m \leftarrow m - \alpha\left[\left(X^\top X + \gamma I\right)m - X^\top y\right]$$

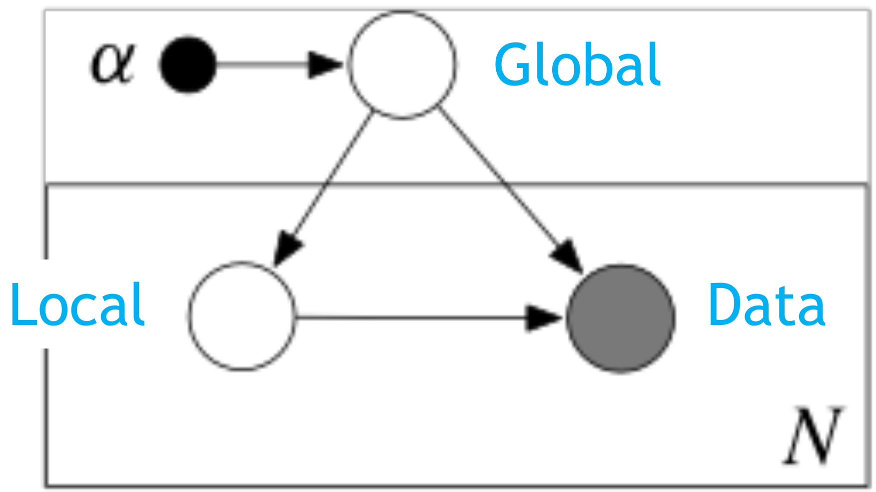
This property generalizes to all "conjugate" models, where forward-backward algorithm returns the natural-gradients of ELBO.

Conditionally-Conjugate Models

VMP: Sequential update with rho =1



SVI: Update local variable with rho=1 and global variable with rho in (0,1)



For CVI, rho can follow any schedule, and updates can be sequential or parallel.

Images taken from Hoffman et al. (2013) and https://www.zybuluo.com/nanmeng/note/369145

Convergence Rates for CVI

Lipschitz constant of (nonconvex) ELBO

Gradient noise variance

$$\mathbb{E}\left[\|(\lambda_k - \lambda_{k+1})/\rho\|^2\right] \leq \left[\frac{2LC_0}{\alpha_*^2 t} + \frac{c\sigma^2}{M\alpha_*}\right]$$

Strong convexity of the Fisher Information Matrix

Mini-batch size

See Khan et al. UAI 2016. The proof is based on Ghadimi, Lan, and Zhang (2014)

NGD for Deep Learning

Using CVI on Bayesian deep learning
with Gaussian approximation.
Reduces to a Newton step.

CVI for Bayesian Neural Network

$$\mathbb{E}_q\left[\sum_{i=1}^{N} \overbrace{\log p(y_i|\underbrace{f_\theta(x_i)}_{\text{neural network}})}^{\text{likelihood}} + \overbrace{\gamma\theta^\top\theta}^{\text{prior}} \quad \overbrace{- \log q_\lambda(\theta)}^{\text{approx}}\right]$$

$$\boxed{\begin{aligned} m &\leftarrow m - \beta(S + \gamma I)^{-1}[g_i(\theta) + \gamma m] \\ S &\leftarrow (1-\beta)S + \beta H_i(\theta) \end{aligned}}$$

Back-propagated gradient & Hessian

$$\theta \sim q_\lambda(\theta), \qquad g_i(\theta) := -\nabla_\theta \log p(y_i|f_\theta(x_i)),$$

$$V^{-1} \leftarrow S + \gamma I, \qquad H_i(\theta) := -\nabla_\theta^2 \log p(y_i|f_\theta(x_i))$$

CVI for Bayesian Neural Network

$$(X^\top X + \gamma I)^{-1} X^\top y$$

$$m \leftarrow m - \beta (S + \gamma I)^{-1} [g_i(\theta) + \gamma m]$$

$$S \leftarrow (1 - \beta) S + \beta H_i(\theta)$$

Back-propagated gradient & Hessian

$$\theta \sim q_\lambda(\theta), \qquad g_i(\theta) := -\nabla_\theta \log p(y_i | f_\theta(x_i)),$$

$$V^{-1} \leftarrow S + \gamma I, \qquad H_i(\theta) := -\nabla^2_\theta \log p(y_i | f_\theta(x_i))$$

MLE vs NGD-VI

RMSprop for MLE

$$\theta \leftarrow \mu$$

$$g \leftarrow \frac{1}{M} \sum_i \nabla_\theta \log p(\mathcal{D}_i|\theta)$$

$$s \leftarrow (1-\beta)s + \beta g^2$$

$$\mu \leftarrow \mu + \alpha \, \frac{g}{\sqrt{s} + \delta}$$

NGD for mean-field VI

$$\theta \leftarrow \mu + \epsilon, \;\; \text{where } \epsilon \sim \mathcal{N}(0, Ns + \lambda)$$

$$g \leftarrow \frac{1}{M} \sum_i \nabla_\theta \log p(\mathcal{D}_i|\theta)$$

$$s \leftarrow (1-\beta)s + \beta g^2$$

$$s \leftarrow (1-\beta)s + \beta \frac{1}{M} \sum_i \nabla^2_{\theta\theta} \log p(\mathcal{D}_i|\theta)$$

$$s \leftarrow (1-\beta)s + \beta \frac{1}{M} \sum_i \left[\nabla_\theta \log p(\mathcal{D}_i|\theta)\right]^2$$

$$\mu \leftarrow \mu + \alpha \, \frac{g + \lambda\mu/N}{\sqrt{s} + \lambda/N}$$

$$\mu \leftarrow \mu + \alpha \, \frac{g + \lambda\mu/N}{s + \lambda/N}$$

Variational Online-Newton (VON)

Variational Online Gauss-Newton (VOGN)

Variational RMSprop (Vprop)

Variational Adam (Vadam)

Adam for MLE

$$\theta \leftarrow \mu$$

$$g \leftarrow \frac{1}{M} \sum_i \nabla_\theta \log p(\mathcal{D}_i|\theta)$$

$$s \leftarrow (1-\beta)s + \beta g^2$$

$$m \leftarrow (1-\gamma)m + \gamma g$$

$$\hat{m} \leftarrow m/(1-(1-\gamma)^t)$$

$$\hat{s} \leftarrow s/(1-(1-\beta)^t)$$

$$\mu \leftarrow \mu + \alpha \, \frac{\hat{m}}{\sqrt{\hat{s}} + \delta}$$

Variational Adam

$$\theta \leftarrow \mu + \epsilon, \text{ where } \mathcal{N}(0, Ns + \lambda)$$

$$g \leftarrow \frac{1}{M} \sum_i \nabla_\theta \log p(\mathcal{D}_i|\theta)$$

$$s \leftarrow (1-\beta)s + \beta g^2$$

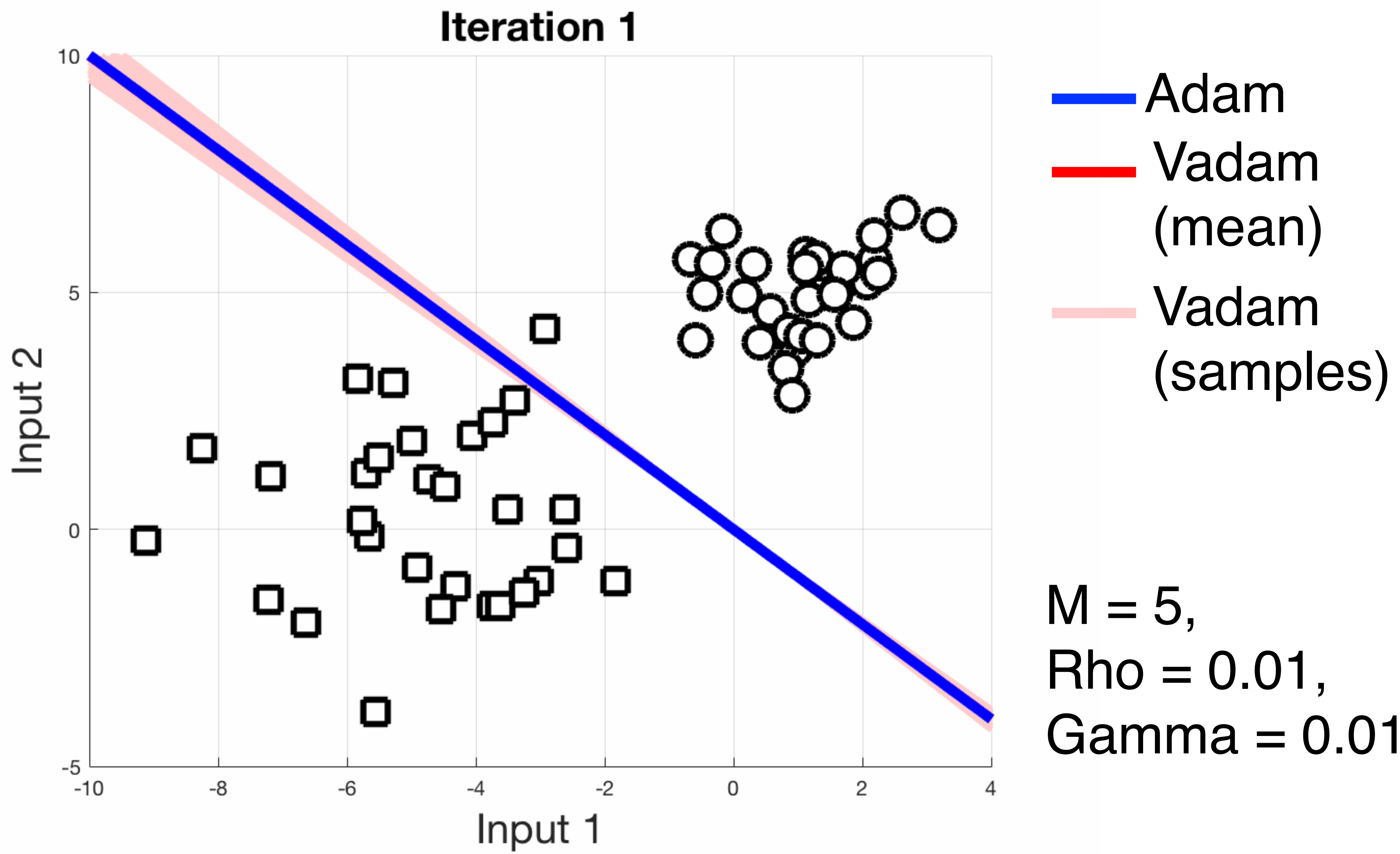
$$m \leftarrow (1-\gamma)m + \gamma(g + \lambda\mu/N)$$

$$\hat{m} \leftarrow m/(1-(1-\gamma)^t)$$

$$\hat{s} \leftarrow s/(1-(1-\beta)^t)$$

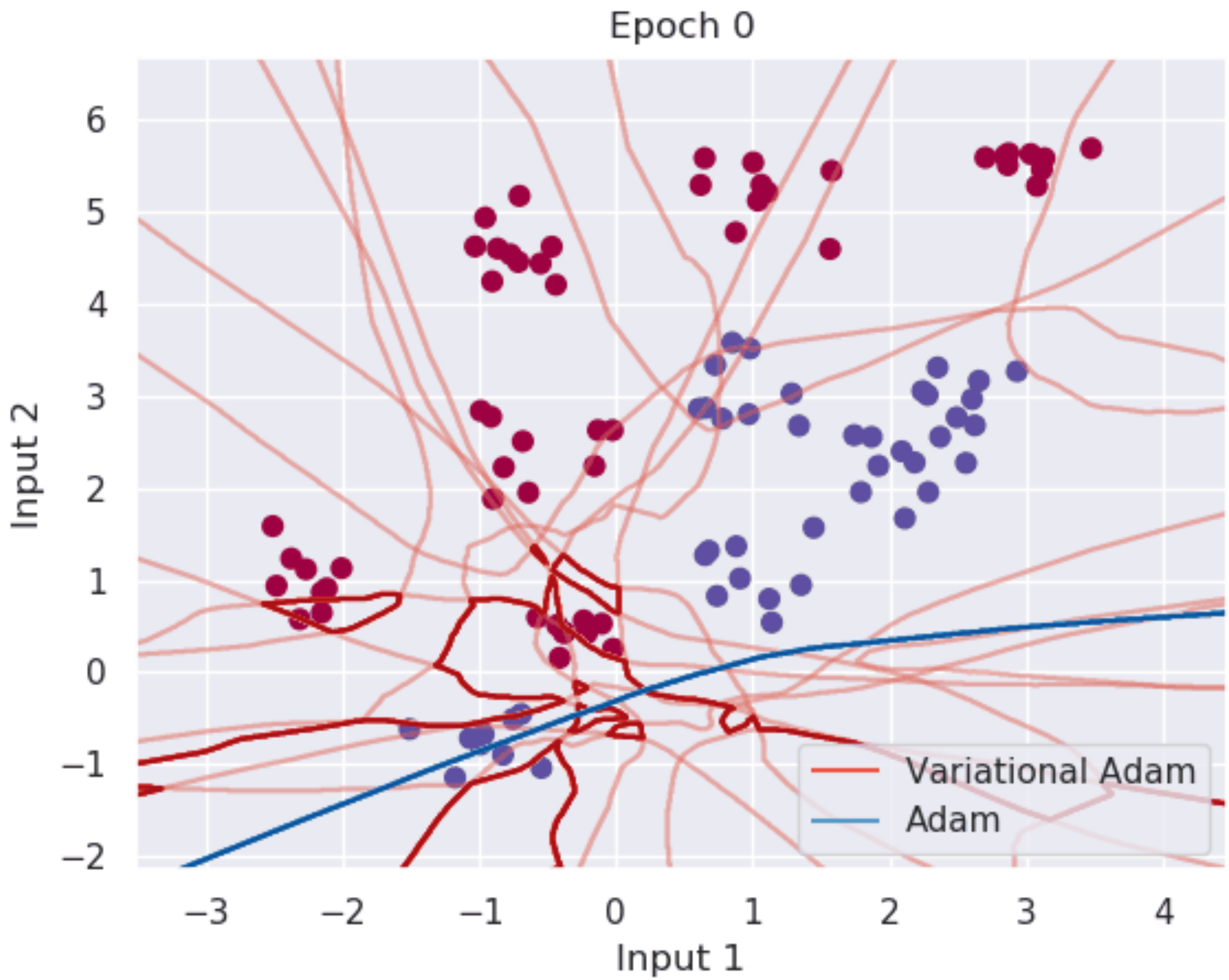
$$\mu \leftarrow \mu + \alpha \, \frac{\hat{m}}{\sqrt{\hat{s}} + \lambda/N}$$

Adam vs Vadam (on Logistic-Reg)



M = 5,
Rho = 0.01,
Gamma = 0.01

Adam vs Vadam (on Neural Nets)



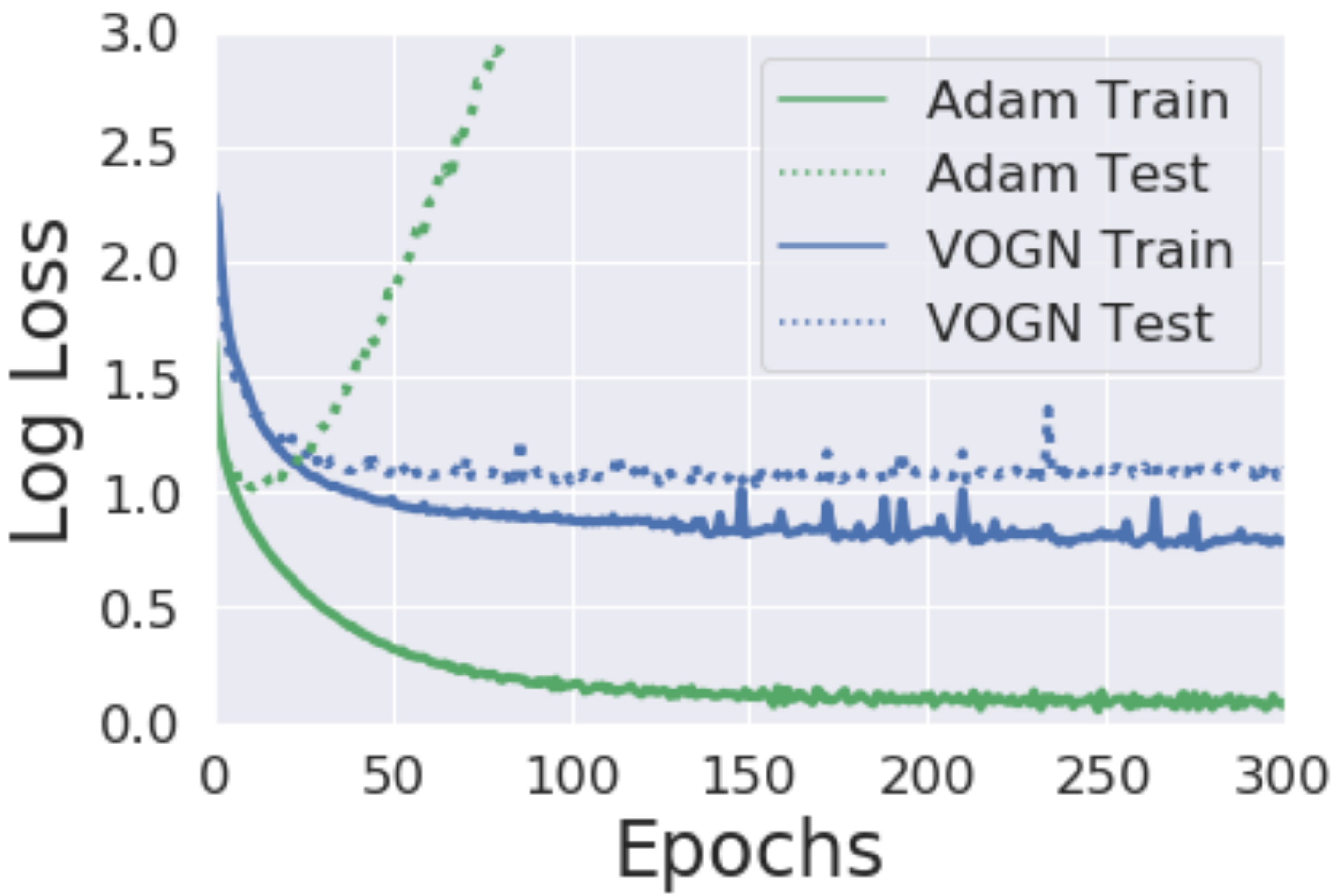
Adam

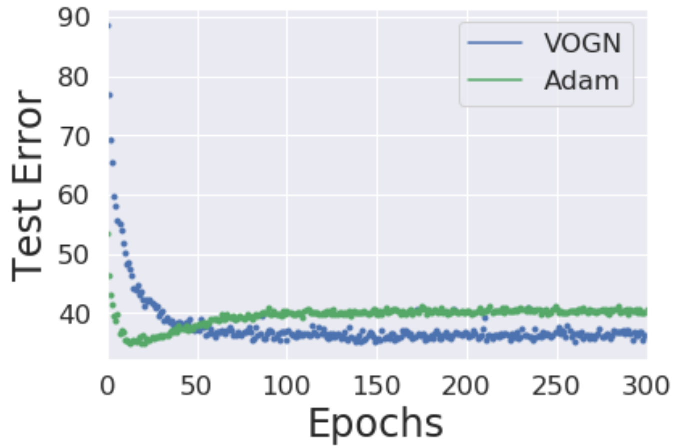
Vadam (mean)

Vadam (samples)

(By Runa E.)

LeNet-5 on CIFAR10





	VOGN	Adam
Log Loss	**1.130**	8.341
Error	**37.01**	40.47

(By Anirudh Jain)

Stochastic, Low-Rank, Approximate, Natural-Gradient (SLANG)

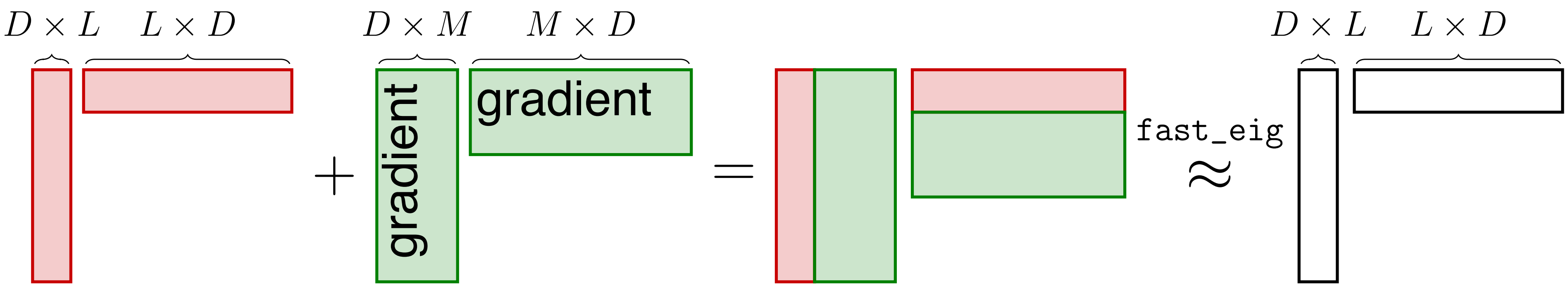
NeurIPS 2018

- Low-rank + diagonal covariance matrix.
- SLANG is linear in D!

Low-Rank + diagonal

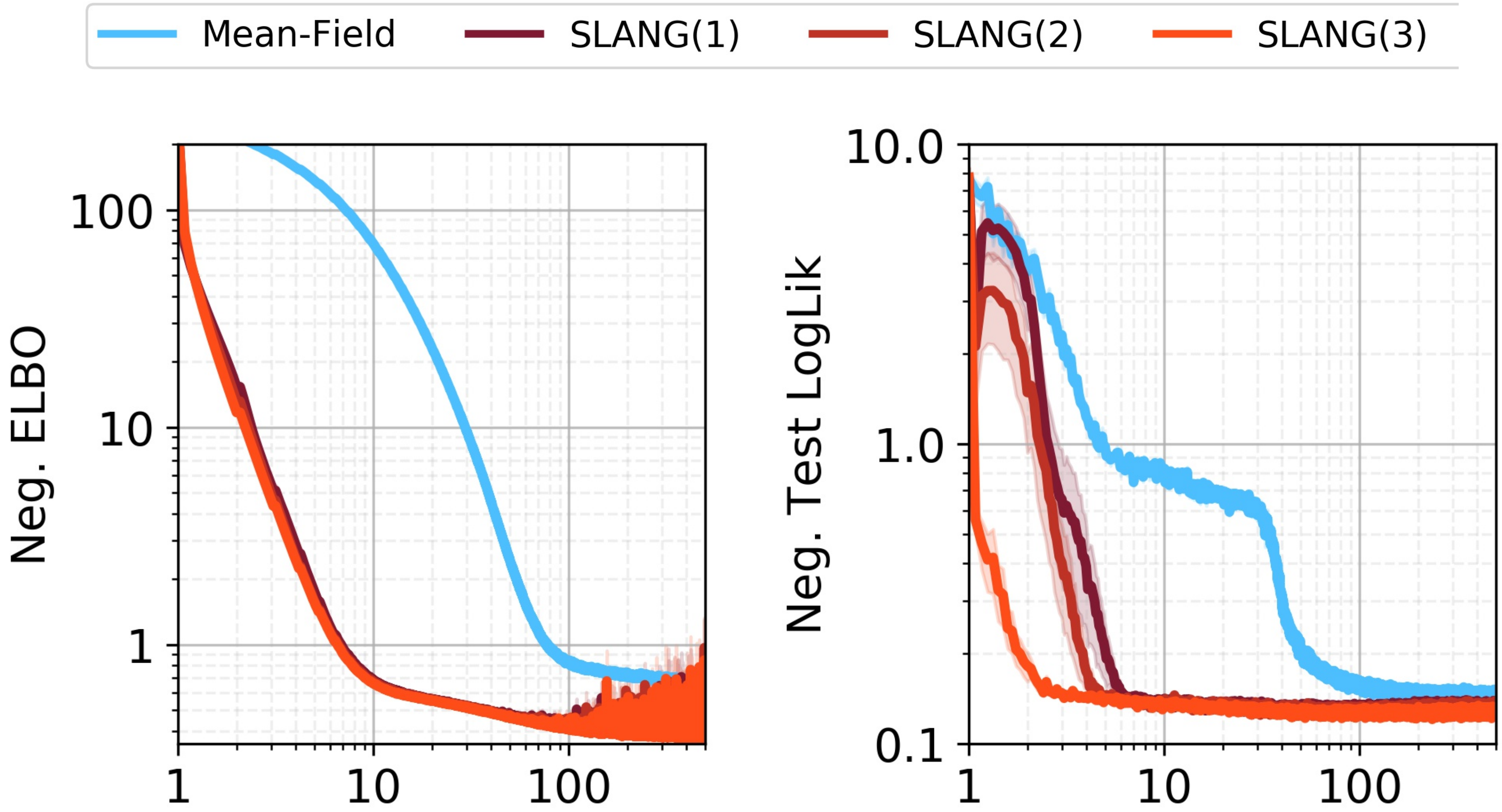
$$m \leftarrow m - \rho\left[UU^\top + D\right]^{-1}\left[g_i + \gamma m\right]$$

$$(1-\beta)S + \beta H_i(\theta)$$



SLANG is Faster than GD

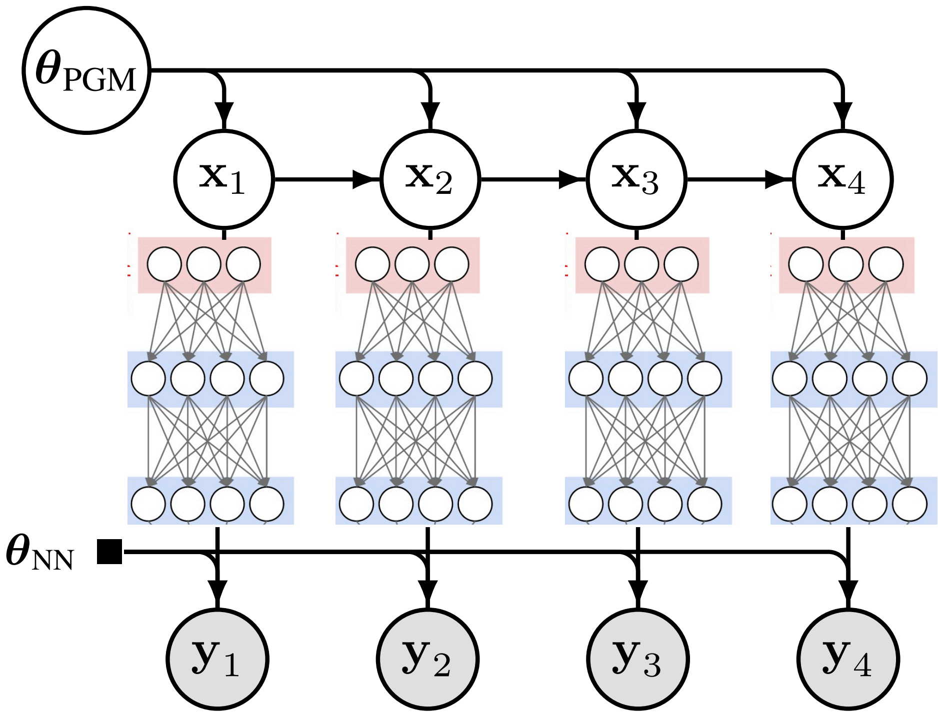
Classification on USPS with BNNs



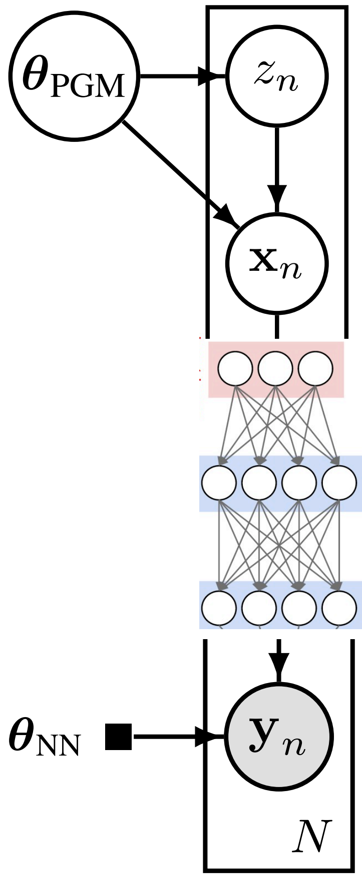
Generalization and Extensions

Deep Nets + Graphical Models

Neural Nets + Linear Dynamical System



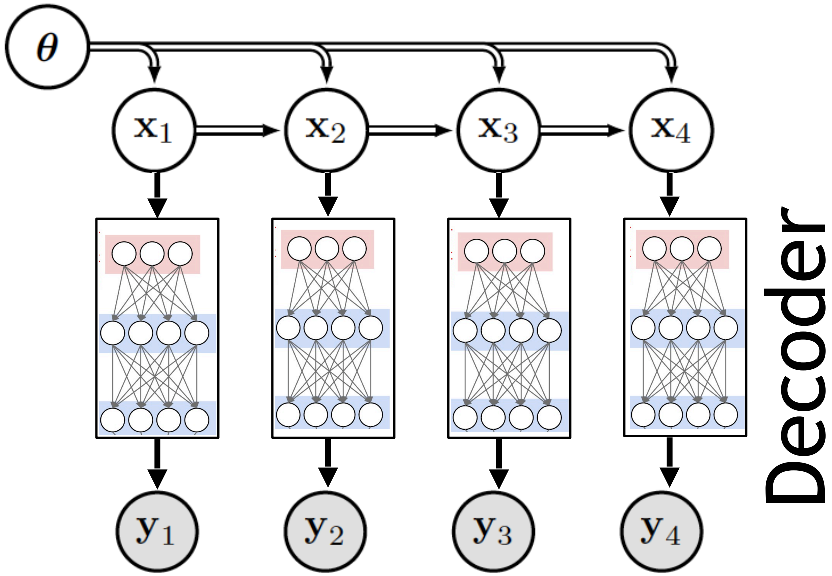
Neural Nets + GMM



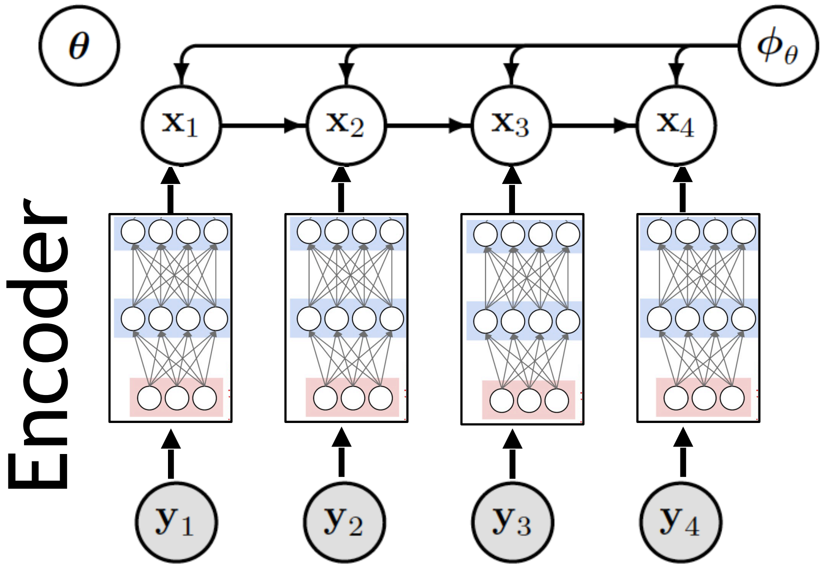
Amortized Inference on VAE + Probabilistic Graphical Models (PGM)

ICLR 2018

Graphical model + Deep Model



Structured Inference Network



Backprop on DNN, and forward-backward on PGM.

Going Beyond Exponential Family

- Fast and Simple NGD for approximations outside exponential family,
  - Scale mixture of Gaussians, e.g., T-distribution,
  - Finite mixture of Gaussian,
  - Matrix Variate Gaussian,
  - Skew-Gaussians.
- The updates can be implemented using message passing and back-propagation.

Summary of the Talk

- Fast yet simple NGD for VI using Conjugate-Computation VI (AI-STATS 2017),
  - Generalization of forward-backward algorithm, Stochastic VI, Variational Message Passing etc.
  - Beyond conjugacy: Extends fast and simple NGD to deep nets (ICML 2018, NeurIPS 2018).
- Generalizations and Extensions,
  - VAEs (ICLR 2018), Mixture of Exponential Family, Evolution strategy (ArXiv 2017), etc.

Related Works

Sorry, if I miss some important work! Please email me.

EM, Forward-Backward, and VI

- Sato (1998), *Fast Learning of On-line EM Algorithm.*
- Sato (2001), *Online Model Selection Based on the Variational Bayes*.
- Jordan et al. (1999), *An Introduction to Variational Methods for Graphical Models*.
- Winn and Bishop (2005), *Variational Message Passing*.
- Knowles and Minka (2011), *Non-conjugate Variational Message Passing for Multinomial and Binary Regression*.

NGD: Author Name Starting with an H

- Honkela et al. (2007), *Natural Conjugate Gradient in Variational Inference*.
- Honkela et al. (2010), *Approximate Riemannian Conjugate Gradient Learning for Fixed-Form Variational Bayes*.
- Hensman et al. (2012), *Fast Variational Inference in the Conjugate Exponential Family*.
- Hoffman et al. (2013), *Stochastic Variational Inference*.

NGD: Author Name Starting with an S

- Salimans and Knowles (2013), *Fixed-Form Variational Posterior Approximation through Stochastic Linear Regression*.
  - Approximate Natural-Gradient steps.
- Seth and Khardon (2016), *Monte Carlo Structured SVI for Two-Level Non-Conjugate Models*.
  - Applies to models with two level of hierarchy.
- Salimbani et al. (2018), *Natural Gradients in Practice: Non-Conjugate Variational Inference in Gaussian Process Models.*
  - Fast convergence on GP models

NGD for Bayesian Deep Learning

- Zhang et al. (2018), *Noisy Natural Gradient as Variational Inference*
  - For Bayesian deep learning (similar to Variational Adam).

Issues and Open Problems

- Automatic natural-gradient computation.
- Good implementation of message passing.
  - Gradient with respect to covariance matrices.
- Structured approximation for covariance.
- Comparisons on really large problems.
- Applications.
- Flexible posterior approximations.

References

Available at https://emtiyaz.github.io/publications.html

Conjugate-Computation Variational Inference : Converting Variational Inference in Non-Conjugate Models to Inferences in Conjugate Models, (AIstats 2017) **M.E. Khan** and W. Lin [Paper] [Code

Faster Stochastic Variational Inference using Proximal-Gradient Methods with General Divergence Functions, (UAI 2016) **M.E. Khan**, R. Babanezhad, W. Lin, M. Schmidt, M. Sugiyama [Paper + Appendix] [Code]

References

Available at https://emtiyaz.github.io/publications.html

Variational Message Passing with Structured Inference Networks,
(ICLR 2018) W. Lin, N. Hubacher, and **M.E. Khan**, [Paper] [ArXiv Version]

Fast and Scalable Bayesian Deep Learning by Weight-Perturbation in Adam,
(ICML 2018) **M.E. Khan**, D. Nielsen, V. Tangkaratt, W. Lin, Y. Gal, and A. Srivastava, [ArXiv Version] [Code] [Slides]

Fast yet Simple Natural-Gradient Descent for Variational Inference in Complex Models,
Invited paper at (ISITA 2018) **M.E. Khan** and D. Nielsen, [Pre-print]

SLANG: Fast Structured Covariance Approximations for Bayesian Deep Learning with Natural Gradient,
(NIPS 2018) A. Miskin, F. Kunstner, D. Nielsen, M. Schmidt, **M.E. Khan**.

Fast and Simple Natural-Gradient Variatioinal Inference with Mixture of Exponential Family,
(Under Submission) W. Lin, M. Schmidt, **M.E. Khan**.

A 5 page review

Fast yet Simple Natural-Gradient Descent for Variational Inference in Complex Models

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***Abstract*—Bayesian inference plays an important role in advancing machine learning, but faces computational challenges when applied to complex models such as deep neural networks. Variational inference circumvents these challenges by formulating Bayesian inference as an optimization problem and solving it using gradient-based optimization. In this paper, we argue in favor of *natural-gradient* approaches which, unlike their *gradient-based* counterparts, can improve convergence by exploiting the information geometry of the solutions. We show how to derive fast yet simple natural-gradient updates by using a duality associated with exponential-family distributions. An attractive feature of these methods is that, by using natural-gradients, they are able to extract accurate local approximations for individual model components. We summarize recent results for Bayesian deep learning showing the superiority of natural-gradient approaches over their gradient counterparts.**

***Index Terms*—Bayesian inference, variational inference, natural gradients, stochastic gradients, information geometry, exponential-family distributions, nonconjugate models.**

prove the rate of convergence [7]–[9]. Unfortunately, these approaches only apply to a restricted class of models known as *conditionally-conjugate* models, and do not work for non-conjugate models such as Bayesian neural networks.

This paper discusses some recent methods that generalize the use of natural gradients to such large and complex non-conjugate models. We show that, for exponential-family approximations, a duality between their natural and expectation parameter-spaces enables a simple natural-gradient update. The resulting updates are equivalent to a recently proposed method called Conjugate-computation Variational Inference (CVI) [10]. An attractive feature of the method is that it naturally obtains *local* exponential-family approximations for individual model components. We discuss the application of the CVI method to Bayesian neural networks and show some recent results from a recent work [11] demonstrating

Acknowledgement

- RIKEN AIP
  - Wu Lin (now at UBC), Didrik Nielsen (now at DTU), Voot Tangkaratt, Nicolas Hubacher, Masashi Sugiyama, Sunichi-Amari.

- Interns at RIKEN AIP
  - Zuozhu Liu (SUTD, Singapore), Aaron Mishkin (UBC), Frederik Kunstner (EPFL).

- Collaborators
  - Mark Schmidt (UBC), Yarin Gal (University of Oxford), Akash Srivastava (University of Edinburgh), Reza Babanezhad (UBC).

Thanks!

Slides, papers, and code available at

https://emtiyaz.github.io