Fast yet Simple Natural-Gradient Descent for Variational Inference

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Variational Inference (VI)

- Approximate the posterior using optimization.
  - Popular in reinforcement learning, unsupervised learning, online learning, active learning etc.
- We need accurate VI algorithms that are
  - general (apply to many models),
  - scalable (for large data and models),
  - fast (convergence quickly),
  - simple (easy to implement).
- This talk: New algorithms with such features.

Gradient vs Natural-Gradient

- Gradient Descent (GD) methods
  - Rely on Stochastic and automatic gradients.
  - Simple, general, scalable, but can have suboptimal convergence (Practical VI (2011), Black-box VI (2014), Bayes by backprop (2015), ADVI (2015), and many more).
- Natural-Gradient Descent (NGD) methods
  - Fast convergence, but computationally difficult, affecting their simplicity, generality and scalability (Sato (2001), Riemannian CG (2010), Stochastic VI (2013), etc.
- Fast and simple NGD for complex models, such as those containing deep networks.

Outline of the Talk

- VI with gradient and natural-gradient descent.
- NGD with Conjugate-Computation VI (AI-STATS 2017),
  - Generalization of forward-backward algorithm, Stochastic VI, Variational Message Passing etc.
  - Beyond conjugacy: Extends fast and simple NGD to deep nets (ICML 2018, NeurIPS 2018).
- Generalizations and Extensions,
  - VAEs (ICLR 2018), Mixture of Exponential Family (AABI 2018), Evolution strategy (ArXiv 2017), etc.

Variational Inference

Gradient Descent (GD)

Vs

Natural-Gradient Descent (NGD)

VI with Gradient Descent (GD)

Parameters

$$p(\theta|\mathcal{D}) = \frac{p(\mathcal{D}|\theta)p(\theta)}{\int p(\mathcal{D}|\theta)p(\theta)d\theta}$$

Data

Intractable integral

Variational Approximation

$$\approx q_\lambda(\theta) = \text{ExpFamily}(\lambda)$$

Natural parameters

Maximize the Evidence Lower Bound (ELBO):

$$\max_\lambda \mathcal{L}(\lambda) := \mathbb{E}_{q_\lambda}\Big[\log p(\mathcal{D},\theta) - \log q_\lambda(\theta)\Big]$$

Gradient descent (GD) : $\lambda \leftarrow \lambda + \rho \nabla_\lambda \mathcal{L}$

VI with Natural-Gradient Descent

Sato 2001, Honkela et al. 2010, Hoffman et.al. 2013

NGD: $\lambda \leftarrow \lambda + \rho \underbrace{F(\lambda)^{-1} \nabla_\lambda \mathcal{L}}$ Natural Gradient

Fisher Information Matrix (FIM)

$$F(\lambda) := \mathbb{E}_{q_\lambda}\left[\nabla \log q_\lambda(\theta) \nabla \log q_\lambda(\theta)^\top\right]$$

- NGD optimizes in the Riemannian manifold instead of the Euclidean geometry (fast convergence).
- But requires additional computations.
- Can we simplify/reduce this computation?

Can we simplify NGD computation? Yes, by using algorithms such as message passing/ backprop.

Conjugate-Computation VI

Khan and Lin, AI-STATS 2017

The key idea: Expectation Parameters

Expectation/moment /mean parameters $\mu := \mathbb{E}_{q_\lambda}[\underbrace{\phi(\theta)}]$ Sufficient statistics

For Gaussians, it's mean and correlation matrix

$$\mathbb{E}_{q_\lambda}[\theta] = m \qquad \mathbb{E}_{q_\lambda}[\theta\theta^\top] = mm^\top + V$$

A key relationship: $F(\lambda)^{-1}\nabla_\lambda \mathcal{L} = \nabla_\mu \mathcal{L}$

Natural Gradient wrt natural parameter

Gradient wrt expectation parameter

NGD : $\lambda \leftarrow \lambda + \rho \nabla_\mu \mathcal{L}$

Conjugate-Computation VI (CVI)

$$\lambda \leftarrow \lambda + \rho \nabla_\mu \mathcal{L}$$

- In a conjugate model, this is equivalent to simply adding the natural parameters of the factors of a model.
- This is a type of conjugate computation, and enables "simple" updates for complex models.

CVI on Bayesian Linear Regression

$$q_\lambda(\theta) := \mathcal{N}(m, V)$$

likelihood prior approx

$$\mathbb{E}_q \left[\underbrace{(y - X\theta)^\top (y - X\theta)^T}_{-\mathbb{E}_{q_\lambda}[\theta]^\top X^\top y + \mathrm{trace}\left[X^\top X \mathbb{E}_{q_\lambda}[\theta\theta^\top]\right]} + \gamma\theta^\top\theta - \log q_\lambda(\theta) \right]$$

$$\begin{aligned} \nabla_{\mathbb{E}_{q_\lambda}[\theta]} &= \begin{pmatrix} -X^\top y & + & 0 & - & V^{-1}m \\ X^\top X & + & \gamma I & - & V^{-1} \end{pmatrix} \\ \nabla_{\mathbb{E}_{q_\lambda}[\theta\theta^\top]} &= \end{aligned}$$

Expectation params Natural Gradient

CVI == Newton's Method

Conjugate-Compute VI for Bayesian linear regression:

$$m \leftarrow (1-\rho)m - \rho \underbrace{\left[X^\top X + \gamma I\right]^{-1} X^\top y}_{\text{Least-square solution}}$$

For $\rho = 1$, converges in 1 step. It's newton's method.

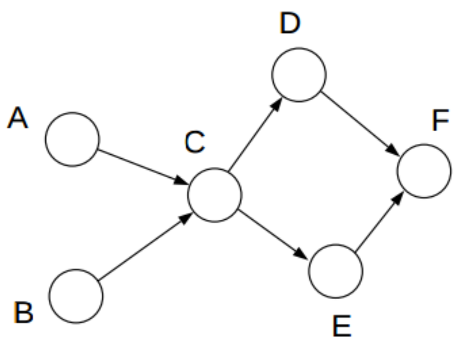
Gradient descent is suboptimal:

$$m \leftarrow m - \alpha\left[\left(X^\top X + \gamma I\right)m - X^\top y\right]$$

This property generalizes to all conjugate models, where forward-backward algorithm returns the natural-gradients of ELBO.

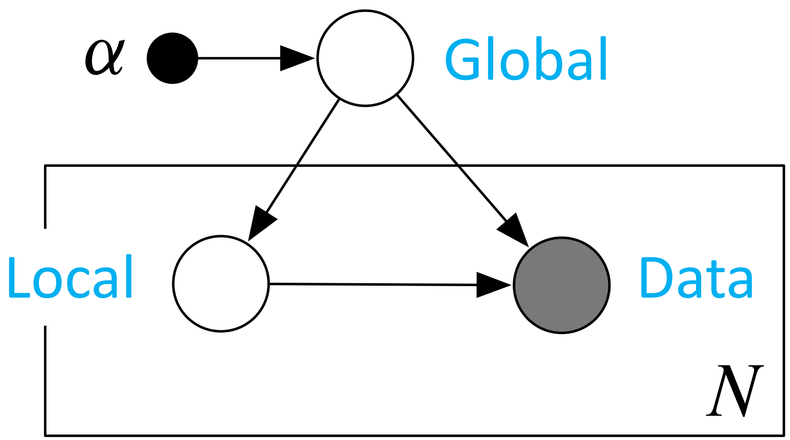
Conditionally-Conjugate Models

- VMP: sequential update with $\rho = 1$



- For CVI, ρ can follow any schedule, and updates can be done sequential or parallel.

- SVI: Update local with $\rho = 1$, then update global with $\rho \in (0,1)$.



Images taken from Hoffman et al. (2013) and https://www.zybuluo.com/nanmeng/note/369145

Convergence Rates for CVI

$$\mathbb{E}_{R,\xi}\left[\|(\boldsymbol{\lambda}_R - \boldsymbol{\lambda}_{R+1})/\beta\|^2\right] \leq \left[\frac{2LC_0}{\alpha_*^2 t} + \frac{c\sigma^2}{M\alpha_*}\right]$$

Lipschitz constant of (nonconvex) ELBO

Gradient noise variance

Strong convexity of the Fisher Information Matrix

Mini-batch size

Based on Ghadimi, Lan and Zhang (2014)

Faster Stochastic Variational Inference using Proximal-Gradient Methods with General Divergence Functions, (UAI 2016) M.E. Khan, R. Babanezhad, W. Lin, M. Schmidt, M. Sugiyama.

NGD for Non-Conjugate Models

Using CVI on Bayesian deep learning
with Gaussian approximation.
Reduces to a Newton step.

CVI for Bayesian Neural Network

$$\mathbb{E}_q \left[\sum_{i=1}^{N} \log p(y_i | f_\theta(x_i)) + \gamma\theta^\top\theta - \log q_\lambda(\theta) \right]$$

likelihood — neural network — prior — approx

$$m \leftarrow m - \beta(S + \gamma I)^{-1}[g_i(\theta) + \gamma m]$$

$$S \leftarrow (1-\beta)S + \beta H_i(\theta)$$

Back-propagated gradient & Hessian

$$\theta \sim q_\lambda(\theta), \qquad g_i(\theta) := -\nabla_\theta \log p(y_i | f_\theta(x_i)),$$

$$V^{-1} \leftarrow S + \gamma I, \qquad H_i(\theta) := -\nabla^2_\theta \log p(y_i | f_\theta(x_i))$$

CVI for Bayesian Neural Network

$$(X^\top X + \gamma I)^{-1} X^\top y$$

$$m \leftarrow m - \beta (S + \gamma I)^{-1} [g_i(\theta) + \gamma m]$$

$$S \leftarrow (1 - \beta) S + \beta H_i(\theta)$$

Back-propagated gradient & Hessian

$$\theta \sim q_\lambda(\theta), \qquad g_i(\theta) := -\nabla_\theta \log p(y_i | f_\theta(x_i)),$$

$$V^{-1} \leftarrow S + \gamma I, \qquad H_i(\theta) := -\nabla^2_\theta \log p(y_i | f_\theta(x_i))$$

Variational Adam for Mean-Field

ICML 2018

Approximate the Hessian by the square of gradients.

Adaptive learning-rate method (e.g., Adam)

Variational Adam (Vadam) for gamma=1

0. Sample ϵ from a standard normal distribution

$$\theta_{\text{temp}} \leftarrow \theta + \epsilon * \underbrace{\sqrt{N * \text{scale} + 1}}_{\text{Variance}}$$

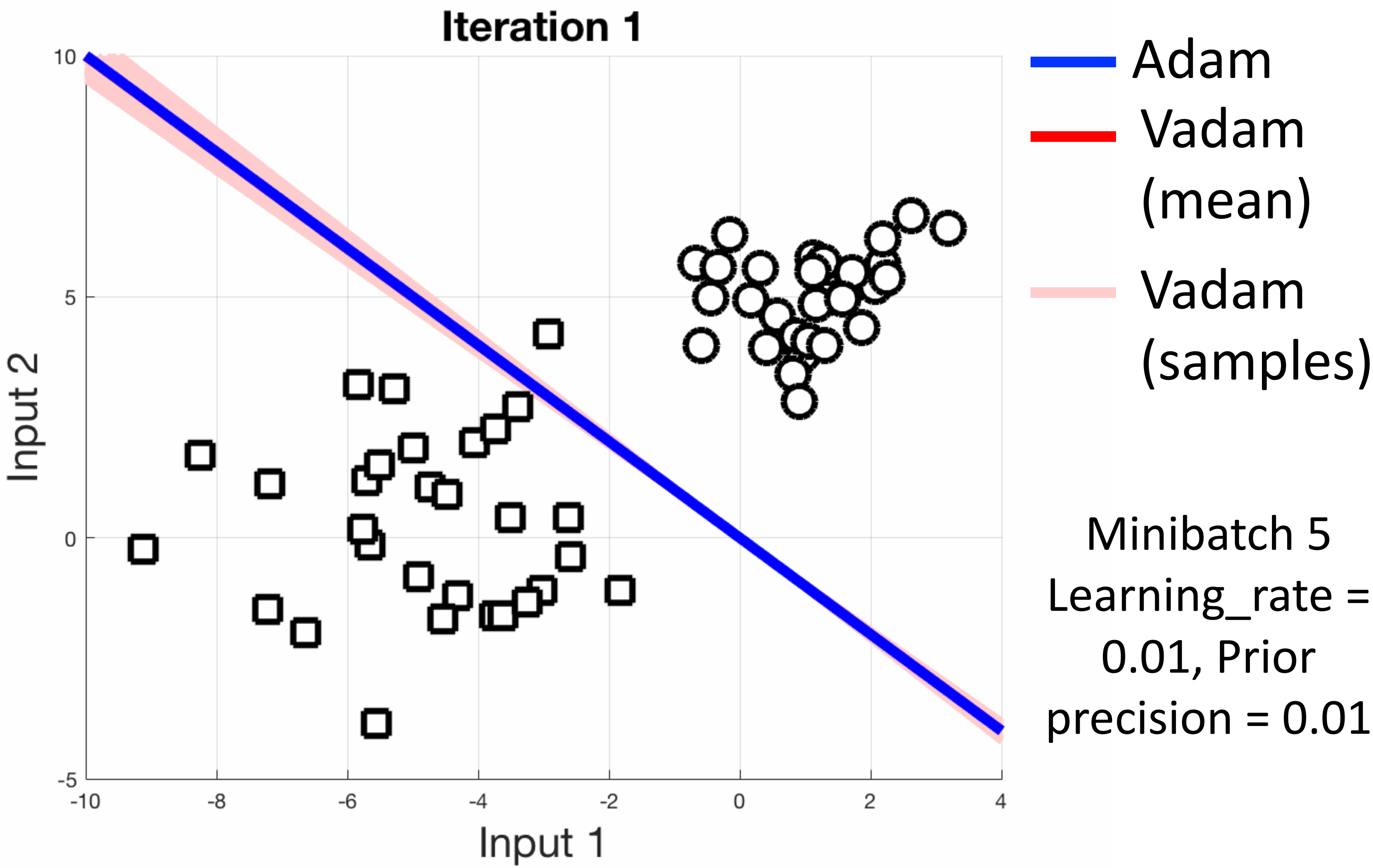
1. Select a minibatch
2. Compute gradient using backpropagation
3. Compute a scale vector to adapt the learning rate
4. Take a gradient step

Mean

$$\theta \leftarrow \theta + \text{learning\_rate} * \frac{\text{gradient}}{\sqrt{\text{scale}} + 10^{-8}}$$

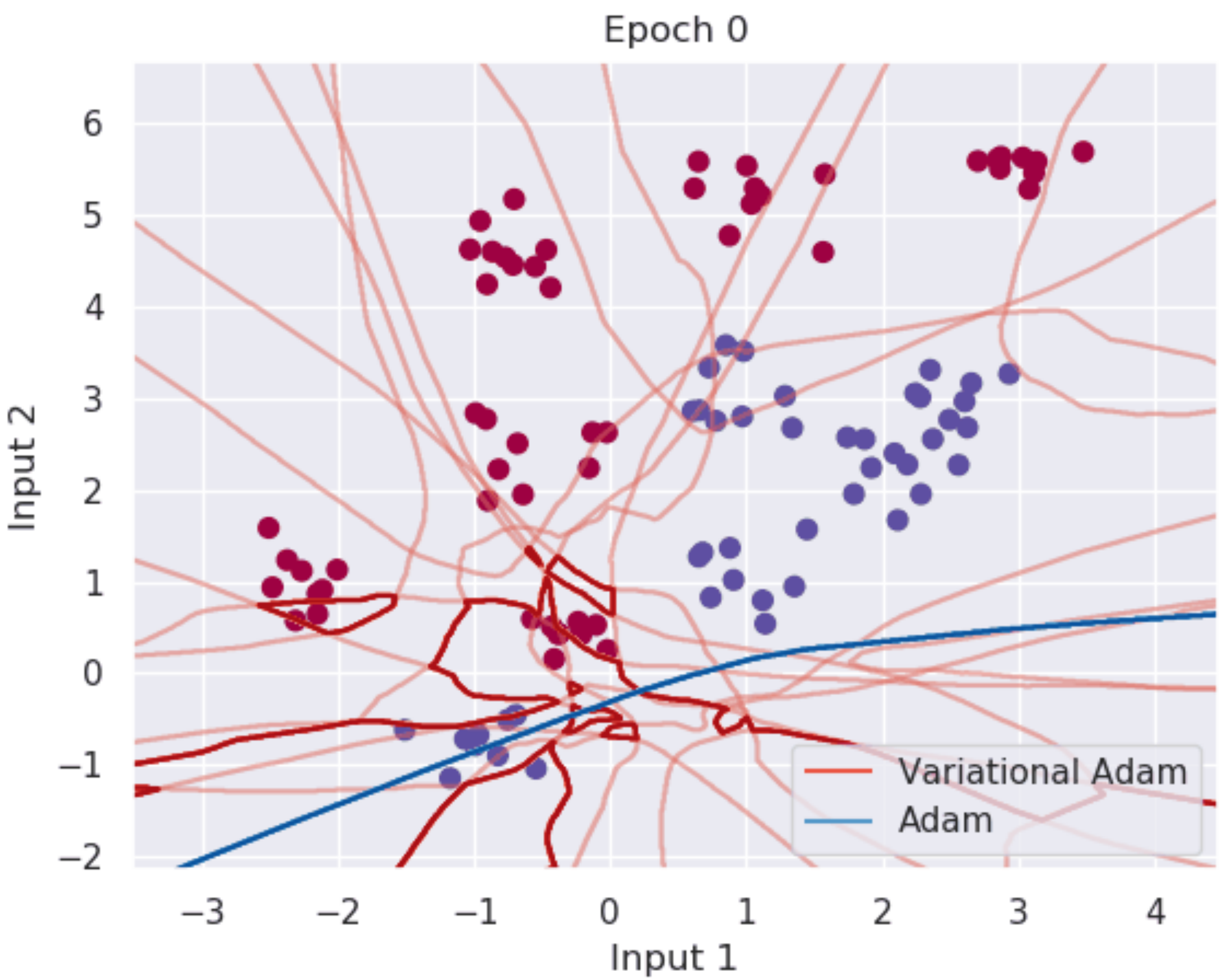
$$\theta \leftarrow \theta + \text{learning\_rate} * \frac{\text{gradient} + \theta/N}{\sqrt{\text{scale}} + 1/N}$$

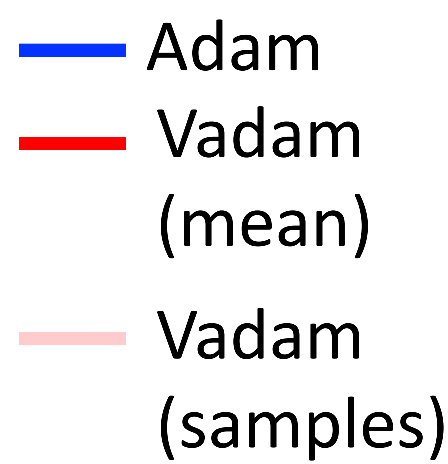
Adam vs Vadam (on Logistic-Reg)



Minibatch 5
Learning_rate = 0.01, Prior precision = 0.01

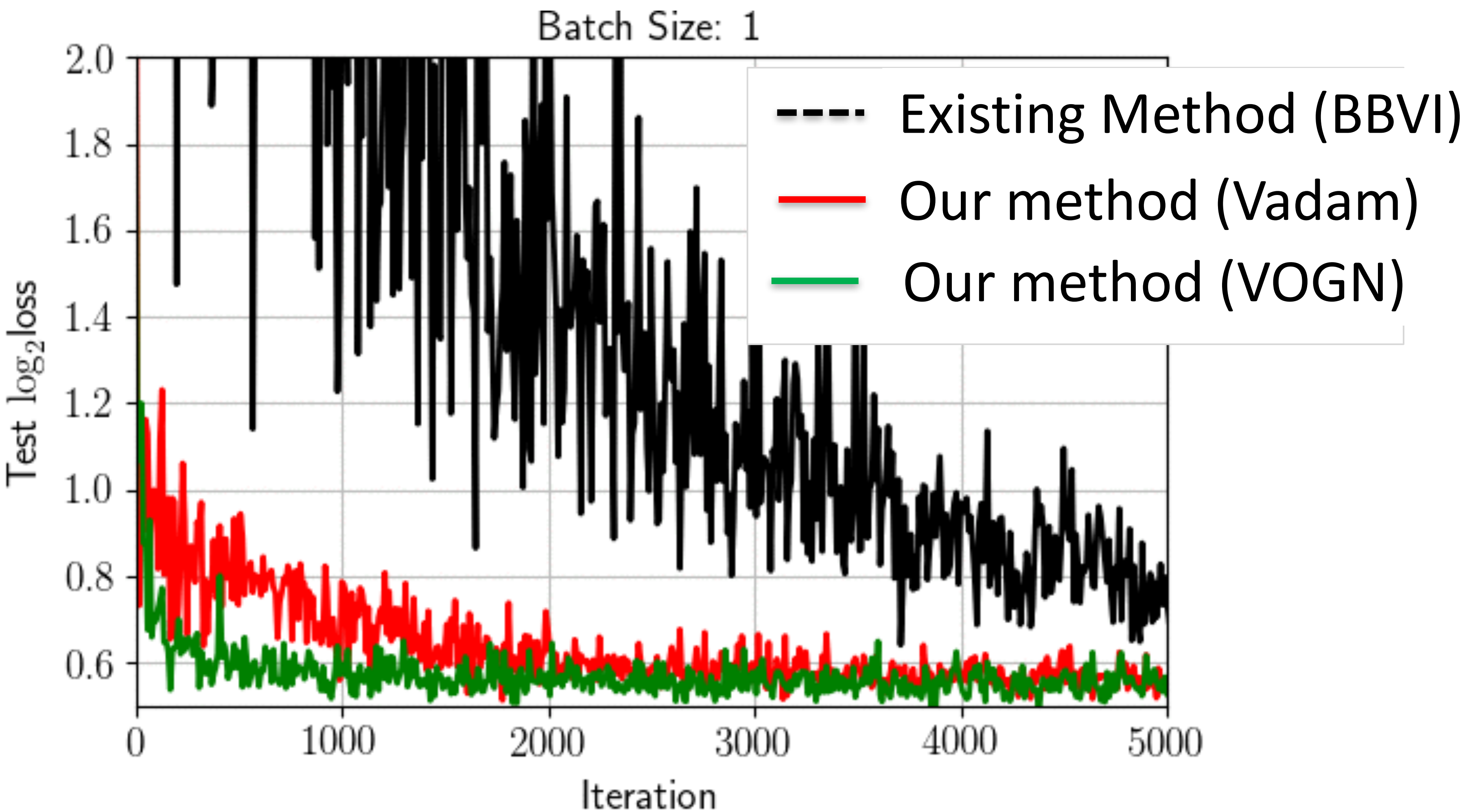
Adam vs Vadam (on Neural Nets)





Faster, Simpler, and More Robust

Regression on Australian-Scale with BNNs

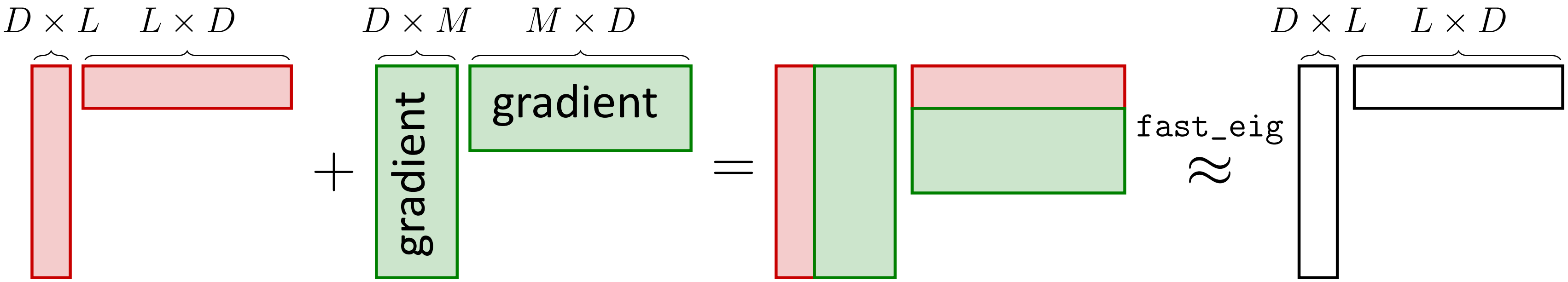


Stochastic, Low-Rank, Approximate, Natural-Gradient (SLANG)

NeurIPS 2018

- Low-rank + diagonal covariance matrix.
- By approximating the Hessian by empirical Fisher.
- SLANG is linear in D!

$$(1-\beta)S + \beta H_i(\theta)$$

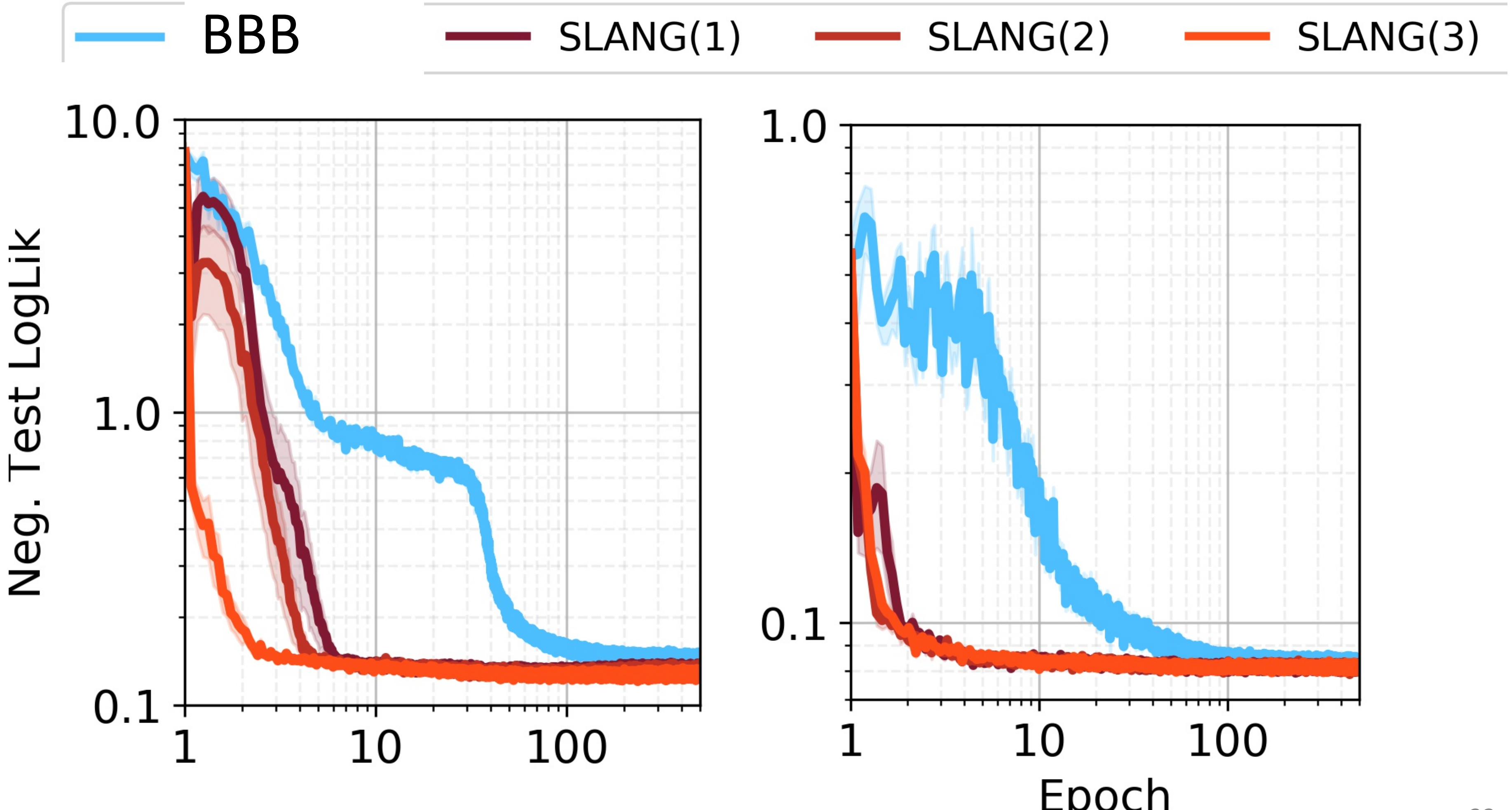


$$\text{SLANG:} \quad \boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t - \alpha_t \left[\mathbf{U}_{t+1}\mathbf{U}_{t+1}^\top + \mathbf{D}_{t+1}\right]^{-1} \left[\hat{\mathbf{g}}(\boldsymbol{\theta}_t) + \lambda\boldsymbol{\mu}_t\right],$$

Low-Rank + diagonal

SLANG is Faster than MF with GD

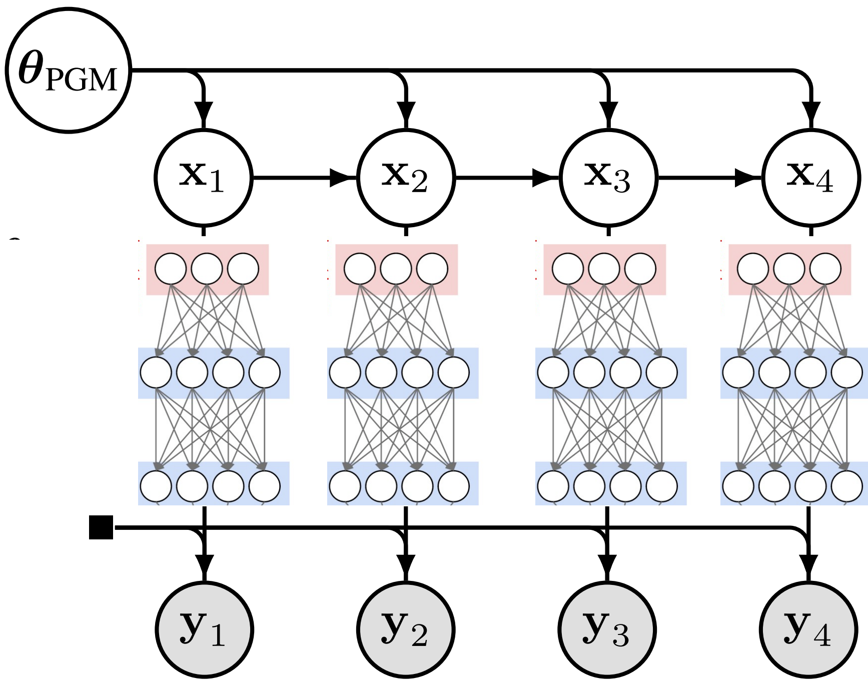
Classification on USPS with BNNs



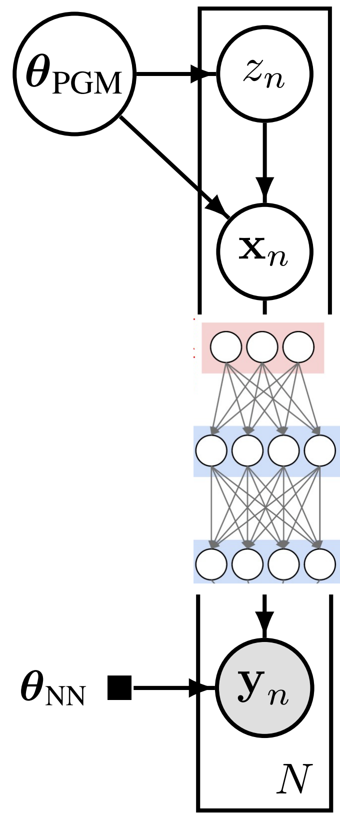
Generalization and Extensions

Deep Nets + Graphical Models

Neural Nets + Linear Dynamical System



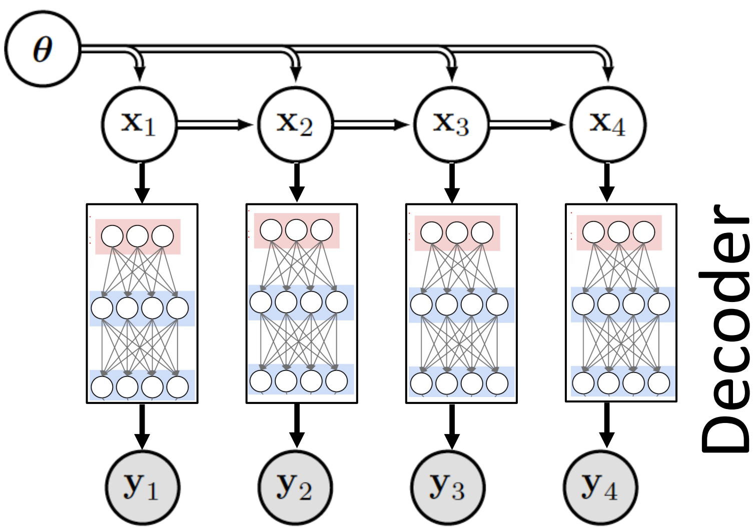
Neural Nets + GMM



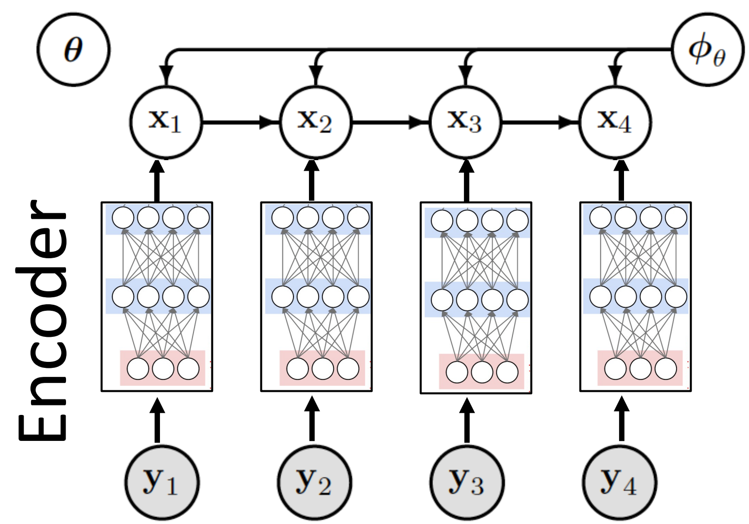
Amortized Inference on VAE + Probabilistic Graphical Models (PGM)

ICLR 2018

Graphical model + Deep Model



Structured Inference Network



Backprop on DNN, and forward-backward on PGM.

Going Beyond Exponential Family

- Fast and Simple NGD for approximations outside exponential family,
  - Scale mixture of Gaussians, e.g., T-distribution,
  - Finite mixture of Gaussian,
  - Matrix Variate Gaussian,
  - Gaussian with Low Rank.
- The updates can be implemented using message passing and back-propagation.

Summary of the Talk

- Fast yet simple NGD for VI using Conjugate-Computation VI (AI-STATS 2017),
  - Generalization of forward-backward algorithm, Stochastic VI, Variational Message Passing etc.
  - Beyond conjugacy: Extends fast and simple NGD to deep nets (ICML 2018, NeurIPS 2018).
- Generalizations and Extensions,
  - VAEs (ICLR 2018), Mixture of Exponential Family (AABI 2018), Evolution strategy (ArXiv 2017), etc.

Related Works

Sorry, if I miss some important work!
Please email me.

EM, Forward-Backward, and VI

- Sato (1998), *Fast Learning of On-line EM Algorithm.*
- Sato (2001), *Online Model Selection Based on the Variational Bayes*.
- Jordan et al. (1999), *An Introduction to Variational Methods for Graphical Models*.
- Winn and Bishop (2005), *Variational Message Passing*.
- Knowles and Minka (2011), *Non-conjugate Variational Message Passing for Multinomial and Binary Regression*.

NGD: Author Name Starting with an H

- Honkela et al. (2007), *Natural Conjugate Gradient in Variational Inference*.
- Honkela et al. (2010), *Approximate Riemannian Conjugate Gradient Learning for Fixed-Form Variational Bayes*.
- Hensman et al. (2012), *Fast Variational Inference in the Conjugate Exponential Family*.
- Hoffman et al. (2013), *Stochastic Variational Inference*.

NGD: Author Name Starting with an S

- Salimans and Knowles (2013), *Fixed-Form Variational Posterior Approximation through Stochastic Linear Regression*.
  - Approximate Natural-Gradient steps.
- Seth and Khardon (2016), *Monte Carlo Structured SVI for Two-Level Non-Conjugate Models*.
  - Applies to models with two level of hierarchy.
- Salimbani et al. (2018), *Natural Gradients in Practice: Non-Conjugate Variational Inference in Gaussian Process Models.*
  - Fast convergence on GP models

NGD for Bayesian Deep Learning

- Zhang et al. (2018), *Noisy Natural Gradient as Variational Inference*
  - For Bayesian deep learning (similar to Variational Adam).

Issues and Open Problems

- Automatic natural-gradient computation.
- Good implementation of message passing.
  - Gradient with respect to covariance matrices.
- Structured approximation for covariance.
- Comparisons on really large problems.
- Applications.
- Flexible posterior approximations.

References

Available at https://emtiyaz.github.io/publications.html

Conjugate-Computation Variational Inference : Converting Variational Inference in Non-Conjugate Models to Inferences in Conjugate Models, (AISTATS 2017) **M.E. KHAN** AND W. LIN [Paper] [Code

Faster Stochastic Variational Inference using Proximal-Gradient Methods with General Divergence Functions, (UAI 2016) **M.E. KHAN**, R. BABANEZHAD, W. LIN, M. SCHMIDT, M. SUGIYAMA [Paper + Appendix] [Code]

References

Available at https://emtiyaz.github.io/publications.html

Variational Message Passing with Structured Inference Networks,
(ICLR 2018) W. LIN, N. HUBACHER, AND **M.E. KHAN**, [Paper] [ArXiv Version]

Fast and Scalable Bayesian Deep Learning by Weight-Perturbation in Adam,
(ICML 2018) **M.E. KHAN**, D. NIELSEN, V. TANGKARATT, W. LIN, Y. GAL, AND A. SRIVASTAVA, [ArXiv Version] [Code] [Slides]

Fast yet Simple Natural-Gradient Descent for Variational Inference in Complex Models,
INVITED PAPER AT (ISITA 2018) **M.E. KHAN** and D. NIELSEN, [Pre-print]

SLANG: Fast Structured Covariance Approximations for Bayesian Deep Learning with Natural Gradient,
(NIPS 2018) A. MISKIN, F. KUNSTNER, D. NIELSEN, M. SCHMIDT, **M.E. KHAN**.

Fast and Simple Natural-Gradient Variatioinal Inference with Mixture of Exponential Family,
(UNDER SUBMISSION) W. LIN, M. SCHMIDT, **M.E. KHAN**.

A 5 page review

Fast yet Simple Natural-Gradient Descent for Variational Inference in Complex Models

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***Abstract*—Bayesian inference plays an important role in advancing machine learning, but faces computational challenges when applied to complex models such as deep neural networks. Variational inference circumvents these challenges by formulating Bayesian inference as an optimization problem and solving it using gradient-based optimization. In this paper, we argue in favor of *natural-gradient* approaches which, unlike their *gradient*-based counterparts, can improve convergence by exploiting the information geometry of the solutions. We show how to derive fast yet simple natural-gradient updates by using a duality associated with exponential-family distributions. An attractive feature of these methods is that, by using natural-gradients, they are able to extract accurate local approximations for individual model components. We summarize recent results for Bayesian deep learning showing the superiority of natural-gradient approaches over their gradient counterparts.**

***Index Terms*—Bayesian inference, variational inference, natural gradients, stochastic gradients, information geometry, exponential-family distributions, nonconjugate models.**

prove the rate of convergence [7]–[9]. Unfortunately, these approaches only apply to a restricted class of models known as *conditionally-conjugate* models, and do not work for non-conjugate models such as Bayesian neural networks.

This paper discusses some recent methods that generalize the use of natural gradients to such large and complex non-conjugate models. We show that, for exponential-family approximations, a duality between their natural and expectation parameter-spaces enables a simple natural-gradient update. The resulting updates are equivalent to a recently proposed method called Conjugate-computation Variational Inference (CVI) [10]. An attractive feature of the method is that it naturally obtains *local* exponential-family approximations for individual model components. We discuss the application of the CVI method to Bayesian neural networks and show some recent results from a recent work [11] demonstrating

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Thanks!

Slides, papers, and code available at

https://emtiyaz.github.io