



### **The Bayesian Learning Rule**

#### Mohammad Emtiyaz Khan RIKEN Center for AI Project, Tokyo http://emtiyaz.github.io



### AI that learn like humans

Quickly adapt to learn new skills, throughout their lives

## Human Learning at the age of 6 months.



# Converged at the age of 12 months



Transfer skills at the age of 14 months



#### Fail because too quick to adapt

#### TayTweets: Microsoft AI bot manipulated into being extreme racist upon release

Posted Fri 25 Mar 2016 at 4:38am, updated Fri 25 Mar 2016 at 9:17am



TayTweets)

https://www.abc.net.au/news/2016-03-25/microsoft-created-ai-bot-becomes-racist/7276266

#### Failure of AI in "dynamic" setting

Robots need quick adaptation to be deployed (for example, at homes for elderly care)



https://www.youtube.com/watch?v=TxobtWAFh8o The video is from 2017

### AI that learn like humans

Quickly adapt to learn new skills, throughout their lives

### Principles of "good" algorithms?

- What are (some) common principles of good algorithms?
- Common origin of Algorithms
  - Revise past belief using new data



### Principles of "good" algorithms?

- Bayesian principles
  - To unify/generalize/improve learning-algorithms
  - By computing "posterior approximations"
- Bayesian Learning rule (BLR)
  - Derive many existing algorithms
  - Deep Learning (SGD, RMSprop, Adam)
  - Design new algorithms for uncertainty in DL
- Impact: Everything with the same principle



#### The Bayesian Learning Rule

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#### Abstract

We show that many machine-learning algorithms are specific instances of a single algorithm called the *Bayesian learning rule*. The rule, derived from Bayesian principles, yields a wide-range of algorithms from fields such as optimization, deep learning, and graphical models. This includes classical algorithms such as ridge regression, Newton's method, and Kalman filter, as well as modern deep-learning algorithms such as stochastic-gradient descent, RMSprop, and Dropout. The key idea in deriving such algorithms is to approximate the posterior using candidate distributions estimated by using natural gradients. Different candidate distributions result in different algorithms and further approximations to natural gradients give rise to variants of those algorithms. Our work not only unifies, generalizes, and improves existing algorithms, but also helps us design new ones.

#### Khan and Rue, The Bayesian Learning Rule, arXiv, https://arxiv.org/abs/2107.04562, 2021

#### Bayesian learning rule

Learning Algorithm	Posterior Approx.	Natural-Gradient Approx.	Sec.			
Optimization Algorithms						
Gradient Descent	Gaussian (fixed cov.)	Delta method	1.3			
Newton's method	Gaussian	"	1.3			
$Multimodal \ optimization \ {}_{\rm (New)}$	Mixture of Gaussians	"	3.2			
Deep-Learning Algorithms						
Stochastic Gradient Descent	Gaussian (fixed cov.)	Delta method, stochastic approx.	4.1			
RMSprop/Adam	Gaussian (diagonal cov.)	Delta method, stochastic approx., Hessian approx., square-root scal- ing, slow-moving scale vectors	4.2			
Dropout	Mixture of Gaussians	Delta method, stochastic approx., responsibility approx.	4.3			
STE	Bernoulli	Delta method, stochastic approx.	4.5			
Online Gauss-Newton (OGN) (New)	Gaussian (diagonal cov.)	Gauss-Newton Hessian approx. in Adam & no square-root scaling	4.4			
Variational OGN (New)	"	Remove delta method from OGN	4.4			
BayesBiNN (New)	Bernoulli	Remove delta method from STE	4.5			
Appro	oximate Bayesian Infere	nce Algorithms				
Conjugate Bayes	Exp-family	Set learning rate $\rho_t = 1$	5.1			
Laplace's method	Gaussian	Delta method	4.4			
Expectation-Maximization	Exp-Family + Gaussian	Delta method for the parameters	5.2			
Stochastic VI (SVI)	Exp-family (mean-field)	Stochastic approx., local $\rho_t = 1$	5.3			
VMP	"	$ \rho_t = 1 $ for all nodes	5.3			
Non-Conjugate VMP	"	"	5.3			
Non-Conjugate VI $_{(New)}$	Mixture of Exp-family	None	5.4			

#### **Principle of Trial-and-Error**

Frequentist: Empirical Risk Minimization (ERM) or Maximum Likelihood Principle, etc.



Deep Learning Algorithms:  $\theta \leftarrow \theta - \rho H_{\theta}^{-1} \nabla_{\theta} \ell(\theta)$ 

Scales well to large data and complex model, and very good performance in practice.

### A Bayesian Origin



#### By changing *Q*, we can recover DL algorithms (and more)

1. Khan and Rue, The Bayesian Learning Rule, arXiv, https://arxiv.org/abs/2107.04562, 2021

2. Khan and Lin. "Conjugate-computation variational inference: Converting variational inference in nonconjugate models to inferences in conjugate models." Alstats (2017).

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### **Bayes Objective**

 $\min_{\theta} \ell(\theta) \quad \text{vs} \quad \min_{q \in \mathcal{Q}} \underbrace{\mathbb{E}_{q(\theta)}[\ell(\theta)] - \mathcal{H}(q)}_{\text{Generalized-Posterior approx.}}$ 



Instead of the original loss, optimize a different (smoothed) one (a popular idea now for DL theory [4]).

A common idea in Inference, optimization, online learning, Reinforcement learning

Zellner, A. "Optimal information processing and Bayes's theorem." *The American Statistician* (1988)
 Many other: Bissiri, et al. (2016), Shawe-Taylor and Williamson (1997), Cesa-Bianchi and Lugosi (2006)
 Huszar's blog, Evolution Strategies, Variational Optimisation and Natural ES (2017)
 Smith et al., On the Origin of Implicit Regularization in Stochastic Gradient Descent, ICLR, 2021

### **Exponential Family**

Natural parameters Sufficient statistics Expectation parameters  

$$q(\theta) \propto \exp \left[\lambda^{\top} T(\theta)\right]$$
 $\downarrow$ 
 $\mu := \mathbb{E}_q[T(\theta)]$ 
 $\mathcal{N}(\theta|m, S^{-1}) \propto \exp \left[-\frac{1}{2}(\theta - m)^{\top} S(\theta - m)\right]$ 
 $\propto \exp \left[(Sm)^{\top} \theta + \operatorname{Tr}\left(-\frac{S}{2}\theta\theta^{\top}\right)\right]$ 

 $\begin{array}{ll} \mbox{Gaussian distribution} & q(\theta) := \mathcal{N}(\theta | m, S^{-1}) \\ \mbox{Natural parameters} & \lambda := \{Sm, -S/2\} \\ \mbox{Expectation parameters} & \mu := \{\mathbb{E}_q(\theta), \mathbb{E}_q(\theta \theta^\top)\} \end{array}$ 

#### **Bayesian learning rule:** $\lambda \leftarrow \lambda - \rho \nabla_{\mu} (\mathbb{E}_q[\ell(\theta)] - \mathcal{H}(q))$

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### **Gradient Descent from Bayes**

Gradient descent:  $\theta \leftarrow \theta - \rho \nabla_{\theta} \ell(\theta)$ 

Bayes Learn Rule:  $m \leftarrow m - \rho \nabla_m \ell(m)$ 

Derived by choosing Gaussian with fixed covariance

 $\begin{array}{ll} \mbox{Gaussian distribution } q(\theta) := \mathcal{N}(m,1) \\ \mbox{Natural parameters} & \lambda := m \\ \mbox{Expectation parameters } \mu := \mathbb{E}_q[\theta] = m \\ \mbox{Entropy} & \mathcal{H}(q) := \log(2\pi)/2 \end{array}$ 

#### **Bayes Prefers Flatter directions**

 $\mathsf{GD:} \quad \theta \leftarrow \theta - \rho \nabla_{\theta} \ell(\theta) \qquad \implies \nabla_{\theta} \ell(\theta_*) = 0$ 

**BLR:**  $m \leftarrow m - \rho \nabla_{\mathbf{m}} \mathbb{E}_q[\ell(\theta)]$ 

$$\implies \nabla_m \mathbb{E}_{q_*}[\ell(\theta)] = 0 \implies \mathbb{E}_{q_*}[\nabla_{\theta} \ell(\theta)] = 0$$

Bayesian solution injects "noise" which has a similar regularization effect to noise in Stochastic GD. It prefers "flatter" directions.



#### **SGD: Implicit Regularization**





#### **SGD: Implicit Regularization**









### **Bayes: Implicit Regularization**

Estimating Gaussian posteriors where the variance is fixed, and only the mean is estimated

 $\mathbb{E}_{q_*}[\nabla_{\theta} \mathcal{E}(\theta)] = 0$ 



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### **Bayes: Implicit Regularization**



### **Bayes: Implicit Regularization**

Bayes solutions (blue) with different variances vs SGD solutions (red lines) with different learning rates.



#### Bayesian learning rule: $\lambda \leftarrow (1 - \rho)\lambda - \rho \nabla_{\mu} \mathbb{E}_q[\ell(\theta)]$

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Put the expectation (Bayes) back in!

The BLR variants [1,2,3] led to the winning solution for the NeurIPS 2021 challenge for "approximate inference in BDL". Watch Thomas Moellenhoff's talk at https://www.youtube.com/ watch?v=LQInIN5EU7E



Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
 Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).

3. Lin et al. "Handling the positive-definite constraints in the BLR." ICML (2020).

See Section 1.3 and 3.2 in Khan and Rue, 2021

#### Deriving Learning-Algorithms from the Bayesian Learning Rule

Posterior Approximation  $\leftrightarrow$  Learning-Algorithm



### **Newton's Method from Bayes**

Newton's method:  $\theta \leftarrow \theta - H_{\theta}^{-1} \left[ \nabla_{\theta} \ell(\theta) \right]$ 

$$Sm \leftarrow (1-\rho)Sm - \rho \nabla_{\mathbb{E}_{q}(\theta)}\mathbb{E}_{q}[\ell(\theta)] \\ -\frac{1}{2}S \leftarrow (1(1-\rho)S)\frac{1}{2}S\rho 2\nabla\rho_{\mathbb{F}_{q}(\theta)}\mathbb{E}_{q}[\ell(\theta)]\theta)]$$

$$\lambda \leftarrow (\lambda 1 - \rho \mathcal{N}_{\mu} \mathcal{H}_{\mu} \mathbb{E}_{q} [\mathcal{H}_{\mu} \mathcal{H}_{\mu}]_{q} [\mathcal{H}_{\mu} \mathcal{H}_{\mu}]_{q} = \lambda$$

Derived by choosing a multivariate Gaussian

 $\begin{array}{ll} \mbox{Gaussian distribution} & q(\theta) := \mathcal{N}(\theta | m, S^{-1}) \\ \mbox{Natural parameters} & \lambda := \{Sm, -S/2\} \\ \mbox{Expectation parameters} & \mu := \{\mathbb{E}_q(\theta), \mathbb{E}_q(\theta \theta^\top)\} \end{array}$ 

1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).

### **Newton's Method from Bayes**

Newton's method:  $\theta \leftarrow \theta - H_{\theta}^{-1} [\nabla_{\theta} \ell(\theta)]$ Set  $\rho = 1$  to get  $m \leftarrow m - H_m^{-1} [\nabla_m \ell(m)]$ 

$$m \leftarrow m - \rho S^{-1} \nabla_m \ell(m)$$
$$S \leftarrow (1 - \rho) S + \rho H_m$$

Delta Method  $\mathbb{E}_q[\ell(\theta)] \approx \ell(m)$ 

Express in terms of gradient and Hessian of loss:  $\nabla_{\mathbb{E}_{q}(\theta)}\mathbb{E}_{q}[\ell(\theta)] = \mathbb{E}_{q}[\nabla_{\theta}\ell(\theta)] - 2\mathbb{E}_{q}[H_{\theta}]m$   $\nabla_{\mathbb{E}_{q}(\theta\theta^{\top})}\mathbb{E}_{q}[\ell(\theta)] = \mathbb{E}_{q}[H_{\theta}]$ 

$$Sm \leftarrow (1-\rho)Sm - \rho \nabla_{\mathbb{E}_{q}(\theta)}\mathbb{E}_{q}[\ell(\theta)]$$
$$S \leftarrow (1-\rho)S - \rho 2\nabla_{\mathbb{E}_{q}(\theta\theta^{\top})}\mathbb{E}_{q}[\ell(\theta)]$$

1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).

#### **Bayes leads to robust solutions**

Avoiding sharp minima



### **Uncertainty of Deep Nets**

#### VOGN: A modification of Adam but match the performance on ImageNet



Code available at <a href="https://github.com/team-approx-bayes/dl-with-bayes">https://github.com/team-approx-bayes/dl-with-bayes</a>

Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
 Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).

### **BLR Variants**

#### RMSprop

Variational Online Gauss-Newton (VOGN)

 $g \leftarrow \hat{\nabla}\ell(\theta)$  $s \leftarrow (1-\rho)s + \rho g^2$  $\theta \leftarrow \theta - \alpha(\sqrt{s} + \delta)^{-1}g$ 

$$g \leftarrow \hat{\nabla}\ell(\theta), \text{ where } \theta \sim \mathcal{N}(m, \sigma^2)$$
$$s \leftarrow (1-\rho)s + \rho(\Sigma_i g_i^2)$$
$$m \leftarrow m - \alpha(s+\gamma)^{-1} \nabla_{\theta}\ell(\theta)$$
$$\sigma^2 \leftarrow (s+\gamma)^{-1}$$

Available at <a href="https://github.com/team-approx-bayes/dl-with-bayes">https://github.com/team-approx-bayes/dl-with-bayes</a>

The BLR variant from [3] led to the winning solution for the NeurIPS 2021 challenge for "approximate inference in deep learning". Watch Thomas Moellenhoff's talk at https://www.youtube.com/watch?v=LQInIN5EU7E.



Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
 Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).
 Lin et al. "Handling the positive-definite constraints in the BLR." ICML (2020).



#### Image Segmentation

Uncertainty (entropy of class probs)

(By Roman Bachmann)<sup>34</sup>



Human Learning at the age of 6 months.

# Contraction Contra

NEURAL INFORMATION MICCISSING STSTEMS

#### by Mohammad Emtiyaz Khan · Dec 9, 2019

#### NeurIPS 2019 Tutorial



by Mohammad Emtiyaz Khan

8.084 views · Dec 9, 2019

Efficient Processing of Deep Neural Network: from Algorithms to...

### **Past and New Work**

#### Natural Gradient Variational Inference

- 1. Khan and Lin. "Conjugate-computation variational inference: Converting variational inference in non-conjugate models to inferences in conjugate models." Alstats (2017).
- 2. Khan and Nielsen. "Fast yet simple natural-gradient descent for variational inference in complex models." (2018) ISITA.

#### • Mixture of Exponential family

3. Lin et al. "Fast and Simple Natural-Gradient Variational Inference with Mixture of Exponential-family Approximations," ICML (2019).

#### Generalization of natural gradients

- 4. Lin et al. "Handling the Positive-Definite Constraint in the Bayesian Learning Rule", ICML (2020)
- 5. Lin et al. "Tractable structured natural gradient descent using local parameterizations", ICML, (2021)
- Gaussian approx ↔ Newton-variants



Wu Lin (UBC)



Mark Schmidt (UBC)



Frank Nielsen (Sony)

### **Gaussian Approximation and DL**

- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
- 2. Mishkin et al. "SLANG: Fast Structured Covariance Approximations for Bayesian Deep Learning with Natural Gradient" NeurIPS (2018).
- 3. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).



#### Extensions

Binary Neural Networks (Bernoulli approx)

1. Meng, et al. "Training Binary Neural Networks using the Bayesian Learning Rule." *ICML* (2020).

Gaussian Process

2. Chang et al. "Fast Variational Learning in State-Space GP Models", MLSP (2020)

- For sparse GPs, BLR is a generalization of [1]





Roman Bachmann (Intern from EPFL)

Xiangming Meng (RIKEN-AIP)







Paul Chang (Aalto University)

W. J. Wilkinson (Aalto University)

Arno Solin (Aalto University)

1. Hensman et al. "Gaussian Process for Big Data", UAI (2013)

#### How to design AI that learn like us?

- Three questions
  - Q1: What do we know? (model)
  - Q2: What do we not know? (uncertainty)
  - Q3: What do we need to know? (action & exploration)
- Posterior approximation is the key
  - (Q1) Models == representation of the world
  - (Q2) Posterior approximations == representation of the model
  - (Q3) Use posterior approximations for knowledge representation, transfer, and collection.

#### **Approximate Bayesian Inference Team**



Emtiyax Khan Team Leader



Pierre Alquier Research Scientist



st Marconi Postdoc



Thomas Möllenhoff Postdoc

#### https://team-approx-bayes.github.io/

We have many open positions! Come, join us.



Lu Xu Postdoc



Jooyeon Kim Postdac



<u>Wu Lin</u> PhD Student University of British Columbia



David Tomàs Cuesta Rotation Student, Okinawa Instituta of Science and Technology



Dharmesh Tallor Remote Collaborator University of Amsterdam



Erik Daxberger Remote Collaborator University of Cambridge



Tojo Rakotoaritina Rotation Student, Okinawa Institute of Science and Technology



Peter Nicki Research Assistant



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Siddharth Swaroop Remote Collaborator University of Cambridge



<u>Alexandre Piché</u> Remote Collaborator *MIL*A



Paul Chang Remote Collaborator Asito University

