State Estimation in Systems with Wireless Devices

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Abstract

A general framework for modeling and analyzing systems with wireless devices is proposed. This framework is used to derive an optimal state estimator when the network introduces random communication delays and packet losses. The framework is general and allows us to analyze earlier results derived in the context of state estimation with delayed and missing observations.

1. INTRODUCTION

Applications such as coordinated control of autonomous vehicles (UAV formations, etc.) and monitoring of plants spread over large areas, involve data transfer over wireless communication links. When compared with wired devices, wireless devices have a number of advantages, such as mobility, flexibility in installation and maintenance, and in many situations, their use is unavoidable. However, constraints inherent to this technology, lead to undesirable effects such as latency and packet losses [1, 2, 3]. To minimize controller performance degradation due to these effects, it is necessary to focus on robustness of control applications in the presence of random delays and missing data.

State estimation is an important component in many model-based, multivariable control techniques and has a direct impact on closed-loop performance. Optimal state estimation techniques are used in a number of signal processing and control applications. The Kalman filter is an optimal, recursive, linear estimator, which estimates the state of a linear system, by weighting the measurements according to a priori information about their accuracies [4]. While, the Kalman Filter was originally developed to deal only with regularly sampled data, it was extended to handle missing data, motivated by multirate applications [5, 6]. State estimation techniques in systems which use wireless devices, were studied to establish statistical convergence properties of the error covariance matrix. Analysis of packet loss effects, led to the establishment of a critical arrival rate of observations, and bounds on the expected state error covariance [2, 3, 7]. Additionally, Smith et. al. [8] used the Jump Markov Linear Systems (JMLS) framework to study these packet loss effects.

State estimation with random delay observations has been addressed in some of the previous works. The JMLS framework was used for state estimation with bounded random delay in [9]. Approaches based on linear matrix inequality and discrete state have been reported in [10, 11, 12]. However all these attempts were made irrespective of the issue of missing observations and an attempt to solve both missing and delayed observations under one framework is missing in the literature.

In the context of state estimation in wireless systems, we can consider two problems to solve, depending on the time instances of missing and delayed observation are known or not for a given observation sequence. We use the term *implementation* problem, to refer to the problem when these are known. On the other hand, in a case where the interest lies in studying the effect of loss and delay probabilities, missing and observation instances may not be known; only the probabilities are available. We term this problem as the *design* problem. In this paper, we have addressed the implementation problem. Work on design problem will be published separately.

In this paper, we propose a stochastic hybrid system framework for analyzing systems which have wireless components. The generality of the framework, allows us to analyze existing results in this area, which were derived under various simplifying assumptions. The rest of the paper is as follows. In Section 2, we present the model based on the event based approach. Following this, in Section 3, we describe derivation of optimal state estimator with innovation approach. In Section 4, we derive the recursive estimators for the missing observation and the one-sample delay cases respectively. Finally, we present a numerical example to demonstrate the use of these estimators in Section 5.

2. MODELING THE NETWORK USING AN EVENT-BASED APPROACH

We assume that the system is as illustrated in the block diagram in Fig. 1. Sensor measurements $\mathbf{y}_t$ from the plant, are communicated through a wireless network channel to give output $\mathbf{z}_t$. We assume that the true plant dynamics are adequately captured by the discrete-time, linear, state-space

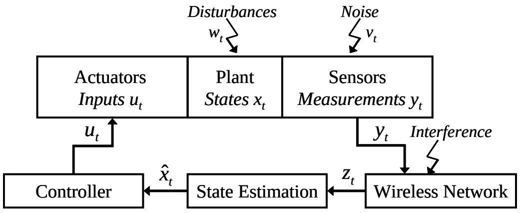

Fig. 1. Block diagram representation of system

model shown in Eq. (1).

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t \\ \mathbf{y}_t &= C\mathbf{x}_t + \mathbf{v}_t \end{aligned} \quad (1)$$

where, $\mathbf{x}_t \in \Re^n$ is the state vector, $\mathbf{y}_t \in \Re^m$ is the output vector, $\mathbf{w}_t \in \Re^n$ and $\mathbf{v}_t \in \Re^m$ are the state and measurement noise vectors respectively. Terms involving the known manipulated inputs are omitted in Eq. (1) because they merely introduce a mean shift in the state-space. We assume that the initial state vector and the noise vectors are *i.i.d* Gaussian random variables, $\mathbf{x}_0 \sim N(\mu_0, \Sigma_0)$, $\mathbf{w}_t \sim N(0,Q)$, $\mathbf{v}_t \sim N(0,R)$, where, Σ_0, Q and R are symmetric, positive definite matrices. For simplicity, $E\left(\mathbf{v}_t\mathbf{w}_t^T\right) = 0$, $E\left(\mathbf{x}_0\mathbf{w}_t^T\right) = 0$ and $E\left(\mathbf{x}_0\mathbf{v}_t^T\right) = 0$, where $E(\,)$ is the *Expectation* operator.

With these assumptions and in the absence of any missing sensor measurements, the Kalman filter is used to compute the minimum mean squared error state estimate [4] for the system represented by Eq. (1). We also assume that the matrix pair, $\{A, Q^{1/2}\}$ is controllable and $\{A, C\}$ is observable. This ensures stability of the Kalman filter.

We define, $\mathbf{Y}_s \equiv \{\mathbf{y}_1, \ldots, \mathbf{y}_s\}$, $\mathbf{Z}_s \equiv \{\mathbf{z}_1, \ldots, \mathbf{z}_s\}$. Further, we use the following definitions for the conditional expectations of the states and the corresponding error covariances: $\hat{\mathbf{x}}_{t|s} = E\left(\mathbf{x}_t|\mathbf{Z}_s\right)$ and $P_{t|s} = E\left((\mathbf{x}_t - \hat{\mathbf{x}}_{t|s})(\mathbf{x}_t - \hat{\mathbf{x}}_{t|s})^T|\mathbf{Z}_s\right)$.

We model the effect of the wireless channel on the sensor measurements using a discrete random variable, F_t, which can take values from the finite set[1], $\mathscr{L}_{F_t} = \{F_{1,t}, F_{2,t}, \ldots, F_{s,t}\}$, at time t. Each of these states represents a different physical event in the network. We use $p_{i,t}$ to denote the probability that $F_t = F_{i,t}$. We make the following assumptions about this discrete random process:

A1 $\mathscr{L}_{F_t}$ is an exhaustive set, *i.e.*, $\sum_{i=1}^{s} p_{i,t} = 1$.

A2 The state-space of F_t is time-invariant, *i.e.*, $F_{i,t} = F^i \;\forall\; i = 1, \ldots, s$. Hence, we drop the time subscript in $\mathscr{L}_{F_t}$.

A3 The vectors, $\mathbf{z}_t$ and $\mathbf{y}_t$ are of the same dimension, *i.e.*, $\mathbf{z}_t \in \Re^m$, and $\mathbf{z}_t$, which is obtained from the wireless channel at time t is an element of the set, $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$.

A4 Measurements obtained from the wireless channel cannot be *out-of-sequence*, *i.e.*, if $\mathbf{z}_i = \mathbf{y}_j, j \le i, \mathbf{z}_{i+c} \notin \{\mathbf{y}_1, \ldots, \mathbf{y}_{j-1}\}$, where c is a positive integer.

We now present a few simple cases to demonstrate our hybrid system representation.

Missing observations: Consider the case of missing observations. In this case the state-space of F can be $\mathscr{L}_F = \{F^1, F^2\}$. If $F_t = F^1$, the observation is available so that $\mathbf{z}_t = \mathbf{y}_t$. In the other case when $F_t = F^2$, no new observation is available from the network. Hence $\mathbf{z}_t = \mathbf{y}_j$, where $\mathbf{y}_j$ is the most recent value successfully communicated through the network (in other words, an old observation in the buffer).

One sample delay: Again the state-space can be $\mathscr{L}_F = \{F^1, F^2\}$, with $F_t = F^1$ for $\mathbf{z}_t = \mathbf{y}_t$ and $F_t = F^2$ for $\mathbf{z}_t = \mathbf{y}_{t-1}$ (delayed by one sample). Note that when the y_t gets delayed at time t, due to our assumptions A3 and A4 we cannot guarantee that $\mathbf{y}_t$ will be observed at the next sampling instant. So if $F_{t+1} = F^2$, $\mathbf{z}_{t+1} = \mathbf{y}_t$, but if $F_{t+1} = F^1$, then $\mathbf{z}_{t+1} = \mathbf{y}_{t+1}$, *i.e.*, the measurement $\mathbf{y}_t$ has been overwritten by $\mathbf{y}_{t+1}$ at the output buffer of the network. Hence, with the proposed framework, *the delay case automatically includes the missing-data case* (similar to the earlier approaches of [10, 11, 12]).

3. STATE ESTIMATION

Estimation objective : *Given observations* $\mathbf{Z}_t$*, find a linear, recursive estimator* $\hat{\mathbf{x}}_{t|t}$ *of* $\mathbf{x}_t$*, which minimizes the trace of the estimation-error covariance matrix* $\mathbf{P}_{t|t}$.

We use *innovation approach* as described in [13]. We will first briefly describe the method of innovation approach for derivation of classical Kalman filter. (In rest of the paper, we assume $\mu_0 = 0$; results obtained can be easily extended to non-zero mean case.). In the absence of wireless links, the observations y_t will be available at all the times. As described in [13], innovation process is based on the orthogonalization procedure, wherein we transform $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t\}$ to an *equivalent* set of orthogonal vectors $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \ldots, \tilde{\mathbf{e}}_t\}$; equivalent in the sense that they span the same linear (sub)space, i.e.,

$$\mathscr{L}\{\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_t\} = \mathscr{L}\{\mathbf{y}_1, \ldots, \mathbf{y}_t\} \quad (2)$$

$\mathscr{L}(S)$ denotes the linear span of set S (different from $\mathscr{L}_F$, state-space of F which is a set). Because of the orthogonality of $\{\tilde{\mathbf{e}}_j\}$, the state estimate $\hat{\mathbf{x}}_{t|t}$ given $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t$ can be found by separately projecting $\mathbf{x}_t$ along each of $\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_t$,

$$\hat{\mathbf{x}}_{t|t} = \sum_{j=1}^{t} \text{Proj}\{\mathbf{x}_t \text{ along } \tilde{\mathbf{e}}_j\}\tilde{\mathbf{e}}_j = \sum_{j=1}^{t} E[\mathbf{x}_t\tilde{\mathbf{e}}_j^T]R_{\tilde{e},j}^{-1}\tilde{\mathbf{e}}_j \quad (3)$$

where $R_{\tilde{e},j} = E[\tilde{\mathbf{e}}_j\tilde{\mathbf{e}}_j^T]$. Here $\text{Proj}\{\mathbf{x}_t \text{ along } \tilde{\mathbf{e}}_j\}$ means 'the projection of $\mathbf{x}_t$ along the orthogonal variable $\mathbf{e}_j$ (refer [13], chapter 4, page 132). The next orthogonal vector corresponding

[1] $\mathscr{L}_{F_t}$ is referred to as "state-space of F_t", and each element is a possible "state of F_t"

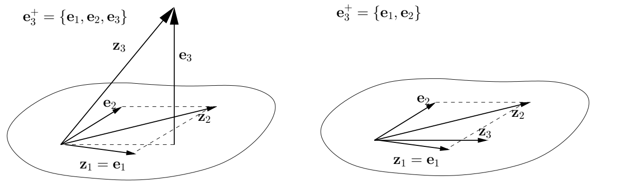

Fig. 2. (Top) When $\mathbf{z}_3 \notin \mathscr{L}\{\mathbf{z}_1,\mathbf{z}_2\}$, a new dimension is added to the subspace, and a non-zero innovation is obtained. $\mathbf{e}_3^+ \equiv \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a basis set for the subspace, (bottom) when $\mathbf{z}_3 \in \mathscr{L}\{\mathbf{z}_1,\mathbf{z}_2\}$, the innovation is undefined, and $\mathbf{e}_3^+ = \mathbf{e}_2^+$.

to the new observation $\mathbf{y}_{t+1}$ can be computed using *Gram-Schmidt orthogonalization* procedure,

$$\tilde{\mathbf{e}}_{t+1} = \mathbf{y}_{t+1} - \sum_{j=1}^{t} E[\mathbf{y}_{t+1}\tilde{\mathbf{e}}_j^T]R_{\tilde{e},j}^{-1}\tilde{\mathbf{e}}_j \tag{4}$$

The orthogonal vector $\tilde{\mathbf{e}}_{t+1}$ can be regarded as "new information" or the "innovation" in $\mathbf{y}_{t+1}$ given $\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_t$, and the process $\{\tilde{\mathbf{e}}_t\}$ as the innovation process associated with $\{\mathbf{y}_t\}$. This formulation can be used to derive the classical Kalman filter recursion [13], which is summarized below:

$$\hat{\mathbf{x}}_{t|t-1} = A\hat{\mathbf{x}}_{t-1|t-1}, \qquad P_{t|t-1} = AP_{t-1|t-1}A^T + Q \tag{5}$$

$$\tilde{\mathbf{e}}_t = \mathbf{y}_t - C\hat{\mathbf{x}}_{t|t-1} \qquad R_{\tilde{e},t} = CP_{t|t-1}C^T + R \tag{6}$$

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + K_t\tilde{\mathbf{e}}_t, \qquad P_{t|t} = P_{t|t-1} - K_tR_{\tilde{e},t}K_t^T \tag{7}$$

where K_t is the Kalman gain and equal to $P_{t|t-1}C^TR_{\tilde{e},t}^{-1}$.

3.1. Innovation approach in the presence of wireless links

As stated in [13] (Chapter 9, page 324), the major assumption made in the earlier described method is that $R_{\tilde{e},j}$ are invertible for all j, which corresponds to a *nondegeneracy assumption* on the process $\{\mathbf{y}_t\}$, *viz.* that no variable $\mathbf{y}_t$ can be estimated without error by some linear combination of earlier variables. Obviously, then, $\mathbf{y}_{t+1} \notin \mathscr{L}\{\mathbf{y}_1,\ldots,\mathbf{y}_t\} = \mathscr{L}\{\tilde{\mathbf{e}}_1,\ldots,\tilde{\mathbf{e}}_t\}$ and hence $\tilde{\mathbf{e}}_{t+1} \neq 0$, and $R_{\tilde{e},t+1}$ is invertible.

However in the presence of wireless links this need not be always true, for e.g. if the observation at time $t+1$ is lost, $\mathbf{z}_t = \mathbf{y}_j$ where $\mathbf{y}_j$ is the most recent value successfully communicated through the network. Hence $\mathbf{z}_{t+1} \in \mathscr{L}\{\mathbf{z}_1,\ldots,\mathbf{z}_t\}$ and no new information will be available in $\mathbf{z}_{t+1}$. In this case innovation will be zero at time $t+1$, and hence it's covariance will not be invertible. This is illustrated in Fig. 2 for $t = 3$.

Hence not all observations will add dimensions to the subspace $\mathscr{L}\{\mathbf{z}_1,\ldots,\mathbf{z}_t\}$ and its dimension can be less than t. So if we have a set of orthogonal vectors[2] $\{\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_t\}$ equivalent to the observations $\{\mathbf{z}_1,\mathbf{z}_2,\ldots,\mathbf{z}_t\}$, only some of the innovations will be orthogonal basis for this subspace (and others will be zero). We denote this set of non-zero innovations as $\mathbf{e}_t^+ \equiv \{\mathbf{e}_j : R_{e,j} > 0, 1 \leq j \leq t\}$, where $R_{e,j} = E[\mathbf{e}_j\mathbf{e}_j^T]$ is covariance of the innovation at time j.

Then the earlier described orthogonalization procedure can be modified by considering only the set $\mathbf{e}_t^+$ for estimation and discarding the other set of innovations[3]. The Eq. (4) will be modified as,

$$\mathbf{e}_{t+1} \quad = \quad \mathbf{z}_{t+1} - \sum_{\mathbf{e}_j \in \mathbf{e}_t^+} E[\mathbf{z}_{t+1}\mathbf{e}_j^T]R_{e,j}^{-1}\mathbf{e}_j \tag{8}$$

Also the state estimates given by Eq. (3) can be rewritten as:

$$\hat{\mathbf{x}}_{t|t} = \sum_{\mathbf{e}_j \in \mathbf{e}_t^+} E[\mathbf{x}_t\mathbf{e}_j^T]R_{e,j}^{-1}\mathbf{e}_j \tag{9}$$

Using this modified innovation approach, we now derive state estimator for different cases.

4. RECURSIVE ESTIMATOR FOR MISSING AND DELAYED OBSERVATION

The events along with the innovations in the case of missing observations are listed here:

F_t	$\mathbf{z}_t$	$\mathbf{e}_t^+$
F^1	$\mathbf{y}_t$	$\{\mathbf{e}_t, \mathbf{e}_{t-1}^+\}$
F^2	$\mathbf{y}_j, j < t$	$\mathbf{e}_{t-1}^+$

We can see that depending on the state of F_t, an innovation will be added to the set $\mathbf{e}_t^+$. Based on this, the state estimator is derived in Appendix A, and summarized here:

Theorem 1 (Recursive estimator for missing observation) *For the missing observation model described in Section 2, the one-step state prediction can be obtained as,*

$$\hat{\mathbf{x}}_{t|t-1} = A\hat{\mathbf{x}}_{t-1|t-1}, \qquad P_{t|t-1} = AP_{t-1|t-1}A^T + Q \tag{10}$$

with $\hat{\mathbf{x}}_{1|0} = 0$ and $P_{1|0} = \Sigma_0$. When observation is received, i.e. $F_t = F^1$, the filtered state estimates are computed by Eq. (7) of the classical Kalman filter. On the other hand for the case of missing observation, we use the prediction $\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1}$ and $P_{t|t} = P_{t|t-1}$.

Note that when $F_t = F^1$, new observation (and hence innovation) is available, and the estimator is similar to the classical Kalman filter recursion. However when $F_t = F^2$, $\mathbf{z}_t \in \mathscr{L}\{\mathbf{z}_1,\ldots,\mathbf{z}_{t-1}\}$ and innovation is zero, and the correction term is missing in the Kalman filter. In [2], missing observations are modeled with an i.i.d. random process γ_t with sample space $\{0,1\}$, on which *pdf* of the observation noise $\mathbf{v}_t$ is conditioned as follows:

$$p(\mathbf{v}_t|\gamma_t) = \begin{cases} \mathscr{N}(0,R), & \text{when } \gamma_t = 1 \\ \mathscr{N}(0,\sigma^2 I), & \text{when } \gamma_t = 0 \end{cases} \tag{11}$$

[2] from now on we refer to $\{\mathbf{e}_t\}$ as innovation process corresponding to $\{\mathbf{z}_t\}$

[3] In other words, we are choosing the weights of the other innovations to be zero

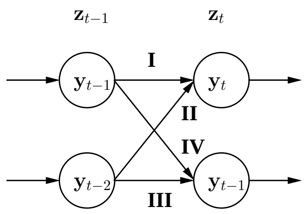

Fig. 3. One sample delay - Four different cases (right) and innovations (left) at time t.

With $\sigma \rightarrow \infty$ the model corresponds to the missing observation. The estimator derived with this approach is same as the proposed estimator. However it seems that the approach of [2] cannot be extended for one sample delay case and it is limited to missing observations only. We now show that the event based approach is quite flexible in this regard and it can be easily extended to the one sample delay case.

The one sample delay case is not as simple as the missing observations, and the four cases involved are shown in Fig. 3. These cases can be described by using four events which are listed in this table:

Case	F_{t-1}	F_t	$\mathbf{e}_t^+$
I	F^1	F^1	$\{\mathbf{e}_t, \mathbf{e}_{t-1}^+\}$
II	F^2	F^1	$\{\mathbf{e}_t, \mathbf{e}_{t-1}^+\}$
III	F^2	F^2	$\{\mathbf{e}_t, \mathbf{e}_{t-1}^+\}$
IV	F^1	F^2	$\mathbf{e}_{t-1}^+$

Except the case IV, in all the others, an innovation will be added at time t to the set $\mathbf{e}_{t-1}^+$. The estimator, derived in Appendix B, is summarized in the following theorem:

Theorem 2 (Recursive estimator for one sample delay case) *The one-step state prediction can be obtained as the classical Kalman filter with Eq. (5). The filtered state estimates are computed for different cases as follows:*

A. *For case I, II with Eq. (7) of classical Kalman filter.*

B. *For case III still use Eq. (7), but with different innovations and Kalman gain computed as*

$$\mathbf{e}_t = \mathbf{z}_t - C\hat{\mathbf{x}}_{t-1|t-1} \quad (12)$$

$$R_{e,t} = CP_{t-1|t-1}C^T + R \quad (13)$$

$$K_t = AP_{t-1|t-1}C^T R_{e,t}^{-1} \quad (14)$$

C. *Case IV use the prediction $\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1}$ and $P_{t|t} = P_{t|t-1}$*

The results are intuitive. In cases I and II, classical Kalman filter is used as the observations are available. Case III is similar to a Kalman filter recursion at time $t-1$, and then a prediction (and hence multiplication by A). Case IV is similar to missing observation as the observation is repeated and no new information is available. A similar estimator has been derived in [12] with innovation approach, however zero innovations have been forced to have invertible covariance, which introduces additional error in estimation. We will show this with simulations in Section 5.

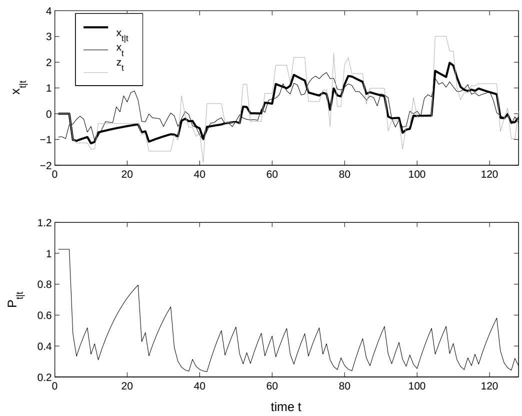

Fig. 4. State estimates $\hat{x}_{t|t}$ and error covariance $P_{t|t}$ for missing observation case.

5. NUMERICAL EXAMPLE

In this section, we present numerical examples to demonstrate the performance of the proposed estimators. We consider the model described in Section 2 with $A = 0.95, C = 1, Q = 0.1, R = 0.9, \mathbf{x}_0 = 0, \Sigma_0 = 1.025641$. Note that this corresponds to a scalar case with $m = n = 1$. Also we have chosen $p_{1,t} = 0.5$.

The Fig. 4 and 5 show the state estimates for missing observation and one sample delay case respectively. We can see that error variance varies with time (unlike the classical Kalman filter). Whenever an observation is lost (or delayed with case IV), innovation is zero and the error variance increases because the correction term is not available then. However it starts decreasing in the other cases.

As stated earlier in Section 4, one sample delay estimator derived in [12] forces zero innovations to have invertible covariance. To show this we compute averaged error covariance $\bar{P}_{t|t}$ for 1000 realizations using both the estimators, the one derived in [12] and our proposed estimator. Fig. 6 shows the comparison, where we can see that because of the proposed modification the error variance has reduced.

6. CONCLUSIONS AND FUTURE WORK

A general framework has been presented for state estimation in systems with wireless devices. Using this framework, optimal, recursive, online state estimators have been developed for the cases where the wireless network introduces random delay and missing observation effects. Preliminary results on simulation case-studies indicate that the state estimates obtained

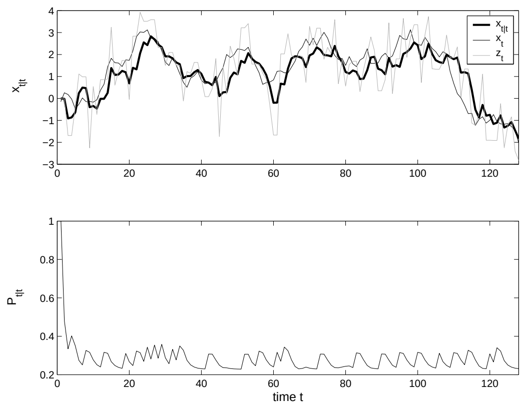

Fig. 5. State estimates $\hat{x}_{t|t}$ and error covariance $P_{t|t}$ for one sample delay observation case.

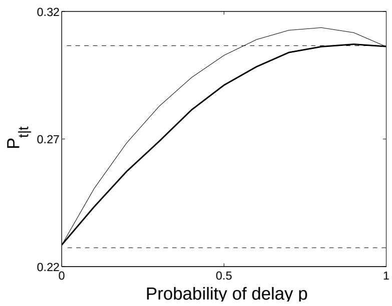

Fig. 6. Comparison of error covariance of the proposed estimator (thick line) and estimator in [12] (thin line).

using this approach have better accuracy in comparison with earlier approaches.

APPENDIX

A. RECURSIVE ESTIMATOR FOR MISSING OBSERVATION

For all $\mathbf{e}_j \in \mathbf{e}_t^+$, $E\left[\mathbf{x}_t \mathbf{e}_j^T\right]$ in Eq. (9) can be expanded using Eq. (8) as follows,

$$E\left[\mathbf{x}_t \mathbf{e}_j^T\right] = E\left[\mathbf{x}_t \mathbf{z}_j^T\right] - \sum_{\mathbf{e}_k \in \mathbf{e}_{j-1}^+} E\left[\mathbf{x}_t \mathbf{e}_k^T\right] R_{e,k}^{-1} E\left[\mathbf{z}_j \mathbf{e}_k^T\right]^T \quad (15)$$

for $2 \leq j \leq t$, and $E\left[\mathbf{x}_t \mathbf{e}_1^T\right] = E\left[\mathbf{x}_t \mathbf{z}_1^T\right]$ for $j = 1$. Also $\mathbf{e}_j \in \mathbf{e}_t^+$ implies $F_j = F^1$, so that $\mathbf{z}_j = \mathbf{y}_j$, and the first term of Eq. (15) can be rewritten as:

$$E\left[\mathbf{x}_t \mathbf{z}_j^T\right] = E\left[\mathbf{x}_t \mathbf{y}_j^T\right] = E\left[\mathbf{x}_t \mathbf{x}_j^T\right] C^T = A^{t-j} R_{x,j} C^T \quad (16)$$

where $R_{x,j} \equiv E[\mathbf{x}_j \mathbf{x}_j^T]$. Using this and Eq. (15), we can see that there exists a function J_j, satisfying,

$$E\left[\mathbf{x}_t \mathbf{e}_j^T\right] = A^{t-j} J_j \quad (17)$$

$$J_j = R_{x,j} C^T - \sum_{\mathbf{e}_k \in \mathbf{e}_{j-1}^+} A^{j-k} J_k R_{e,k}^{-1} E\left[\mathbf{z}_j \mathbf{e}_k^T\right]^T \quad (18)$$

$$J_1 = R_{x,1} C^T \quad (19)$$

Using this in Eq. (9), the state estimates are now given by:

$$\hat{\mathbf{x}}_{t|i} = \sum_{\mathbf{e}_j \in \mathbf{e}_{i-1}^+} A^{t-j} J_j R_{e,j}^{-1} \mathbf{e}_j \quad (20)$$

A.1. Recursions for $\hat{\mathbf{x}}_{t|t-1}$, $P_{t|t-1}$, $\hat{\mathbf{x}}_{t|t}$ and $P_{t|t}$

Using Eq. (20) with $i = t-1$, we get recursion for the one-step state prediction,

$$\hat{\mathbf{x}}_{t|t-1} = \sum_{\mathbf{e}_j \in \mathbf{e}_{t-1}^+} A^{t-j} J_j R_{e,j}^{-1} \mathbf{e}_j = A \hat{\mathbf{x}}_{t-1|t-1} \quad (21)$$

Putting $i = t$ in Eq. (20) we get the filtered estimate,

$$\hat{\mathbf{x}}_{t|t} = \sum_{\mathbf{e}_j \in \mathbf{e}_{t-1}^+} A^{t-j} J_j R_{e,j}^{-1} \mathbf{e}_j + J_t R_{e,t}^{-1} \mathbf{e}_t = \hat{\mathbf{x}}_{t|t-1} + K_t \mathbf{e}_t \quad (22)$$

where K_t is defined as $K_t \equiv J_t R_{e,t}^{-1}$. As expected when the innovation is available, the prediction estimate can be corrected to give filtered estimate. However when $\mathbf{e}_t \notin \mathbf{e}_t^+$, then $\mathbf{e}_t^+ = \mathbf{e}_{t-1}^+$ and hence, $\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1}$.

Next we get the recursion for $P_{t|t-1}$ using the difference of state and estimator covariance matrices (similar approach can be seen in [13], page 328). In the model given by Eq. (1), the covariance matrix of the state-vector follows the recursion,

$$R_{x,t} = A R_{x,t-1} A^T + Q, \qquad R_{x,t} \equiv E[\mathbf{x}_t \mathbf{x}_t^T] \quad (23)$$

Denoting the covariance matrix of one-step state predictor as $\Sigma_{t|t-1} \equiv E\left[\hat{\mathbf{x}}_{t|t-1} \hat{\mathbf{x}}_{t|t-1}^T\right]$ and using the Eq. (21), we have $\Sigma_{t|t-1} = A \Sigma_{t-1|t-1} A^T$ with initial condition $\Sigma_{1|0} = \Sigma_0$. But as $\hat{\mathbf{x}}_{t|t-1}$ is orthogonal to $\mathbf{x}_t - \hat{\mathbf{x}}_{t|t-1}$, and $\mathbf{x}_t = (\mathbf{x}_t - \hat{\mathbf{x}}_{t|t-1}) + \hat{\mathbf{x}}_{t|t-1}$, we get,

$$R_{x,t} = P_{t|t-1} + \Sigma_{t|t-1} \quad (24)$$

so that $P_{t|t-1} = R_{x,t} - \Sigma_{t|t-1} = A P_{t-1|t-1} A^T + Q$. This gives us the recursion for $P_{t|t-1}$.

Similarly, defining the covariance matrix of the filtered state estimator as $\Sigma_{t|t} \equiv E\left[\hat{\mathbf{x}}_{t|t} \hat{\mathbf{x}}_{t|t}^T\right]$, we have,

$$\Sigma_{t|t} = \begin{cases} \Sigma_{t|t-1} + K_t R_{e,t} K_t^T, & \text{for } \mathbf{e}_t \in \mathbf{e}_t^+ \\ \Sigma_{t|t-1}, & \text{for } \mathbf{e}_t \notin \mathbf{e}_t^+ \end{cases} \quad (25)$$

and similar to Eq. (24), we have $R_{x,t} = P_{t|t} + \Sigma_{t|t}$, using which we get recursion for $P_{t|t}$:

$$P_{t|t} = \begin{cases} R_{x,t} - \Sigma_{t|t} = P_{t|t-1} + K_t R_{e,t} K_t^T, & \text{for } \mathbf{e}_t \in \mathbf{e}_t^+ \\ P_{t|t-1}, & \text{for } \mathbf{e}_t \notin \mathbf{e}_t^+ \end{cases} \quad (26)$$

Note that (see table in missing observation case), $\mathbf{e}_t \in \mathbf{e}_t^+$ when $F_t = F^1$ and $\mathbf{e}_t \notin \mathbf{e}_t^+$ when $F_t = F^2$ which gives the corresponding equation for filtering and prediction of Theorem 1.

A.2. Expressions for $\mathbf{e}_t$ and $R_{e,j}$

For all $\mathbf{e}_j \in \mathbf{e}_{t-1}^+$, we can write,

$$E[\mathbf{z}_t \mathbf{e}_j^T] = E[\mathbf{y}_t \mathbf{e}_j^T] = CE[\mathbf{x}_t \mathbf{e}_j^T] + E[\mathbf{v}_t \mathbf{e}_j^T] = CA^{t-j}J_j \quad (27)$$

Substituting this into Eq. (8) and using Eq. (21) we get expression for innovation at time t: $\mathbf{e}_t = \mathbf{z}_t - C\hat{\mathbf{x}}_{t|t-1}$.

Using the fact that $\mathbf{e}_t$ is orthogonal to the past innovation variable, we can write an expression for $R_{e,t}$ using Eq. (8):

$$R_{e,t} = E[\mathbf{e}_t \mathbf{e}_t^T] = E[\mathbf{z}_t \mathbf{z}_t^T] - \sum_{\mathbf{e}_j \notin \mathbf{e}_{t-1}^+} E\left[\mathbf{z}_t \mathbf{e}_j^T\right] R_{e,j}^{-1} E\left[\mathbf{z}_t \mathbf{e}_j^T\right]^T \quad (28)$$

with $R_{e,1} = E[\mathbf{z}_1 \mathbf{z}_1^T]$. We compute $E[\mathbf{z}_t \mathbf{z}_t^T]$ as follows:

$$E[\mathbf{z}_t \mathbf{z}_t^T] = E[\mathbf{y}_t \mathbf{y}_t^T] = CE[\mathbf{x}_t \mathbf{x}_t^T]C^T + R = CR_{x,t}C^T + R \quad (29)$$

Substituting it into Eq. (28) along with $E[\mathbf{z}_t \mathbf{e}_j^T]$ from Eq. (27):

$$R_{e,t} = CR_{x,t}C^T + R - \sum_{\mathbf{e}_j \notin \mathbf{e}_{t-1}^+} CA^{t-j}J_j R_{e,j}^{-1} J_j^T (A^{t-j})^T C^T \quad (30)$$

From Eq. (21), we note,

$$\Sigma_{t|t-1} = E\left[\hat{\mathbf{x}}_{t|t-1}\hat{\mathbf{x}}_{t|t-1}^T\right] = \sum_{\mathbf{e}_j \notin \mathbf{e}_{t-1}^+} A^{t-j}J_j R_{e,j}^{-1} J_j^T (A^{t-j})^T \quad (31)$$

Using this in Eq. (30),

$$R_{e,t} = CR_{x,t}C^T + R - C\Sigma_{t|t-1}C^T = CP_{t|t-1}C^T + R \quad (32)$$

where the last step is by using Eq. (24).

A.3. Expression for K_t

As $K_t = J_t R_{e,t}^{-1}$, we first compute J_t. Eq. (18) can be rewritten:

$$\begin{aligned} J_t &= R_{x,t}C^T - \sum_{\mathbf{e}_j \in \mathbf{e}_{t-1}^+} A^{t-j}J_j R_{e,j}^{-1} J_j (A^{t-j})^T C^T \quad (33) \\ &= R_{x,t}C^T - \Sigma_{t|t-1}C^T = P_{t|t-1}C^T \quad (34) \end{aligned}$$

so that, $K_t = P_{t|t-1}C^T R_{e,t}^{-1}$.

B. RECURSIVE ESTIMATOR FOR ONE SAMPLE DELAY CASE

For the case IV, as shown in Fig. 3, $\mathbf{e}_t \notin \mathbf{e}_t^+$. So the estimator will be similar to classical Kalman filter without correction term. In cases I and II, we have $\mathbf{z_t} = \mathbf{y}_t$, hence the estimator will be the same as the missing observation case when $F_t = F^1$. In case III, as $\mathbf{z_t} = \mathbf{y}_{t-1}$, the estimator can be derived just by replacing $\mathbf{y}_t$ by $\mathbf{y}_{t-1}$ in the missing observation case to get:

$$\begin{aligned} \mathbf{e}_t &= \mathbf{z}_t - C\hat{\mathbf{x}}_{t-1|t-1} \quad (35) \\ R_{e,t} &= CP_{t-1|t-1}C^T + R \quad (36) \\ K_t &= AP_{t-1|t-1}C^T R_{e,t}^{-1} \quad (37) \end{aligned}$$

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