



Bayesian Learning Rule

Mohammad Emtiyaz Khan

RIKEN Center for AI Project, Tokyo

http://emtiyaz.github.io



^{1.} Summary at https://emtiyaz.github.io/papers/MLfromBayes.pdf

Al that learn like humans

Quickly adapt to learn new skills, throughout their lives

Human Learning at the age of 6 months.



Converged at the age of 12 months



Transfer skills at the age of 14 months



Failure of AI in "dynamic" setting

Robots need quick adaptation to be deployed (for example, at homes for elderly care)



Bayesian Principles





Human learning



Deep learning

Life-long learning from small chunks of data in a non-stationary world

Bulk learning from a large amount of data in a stationary world

Our current research focuses on reducing this gap!

- 1. Parisi, German I., et al. "Continual lifelong learning with neural networks: A review." *Neural Networks* (2019)
- 2. Geisler, W. S., and Randy L. D. "Bayesian natural selection and the evolution of perceptual systems." *Philosophical Transactions of the Royal Society of London. Biological Sciences* (2002)

Bayesian Learning Rule

- Bayesian principles as a general principle
 - To unify/generalize/improve learning-algorithms
 - By computing "posterior approximations"
- Bayesian Learning rule (BLR)
 - Derive many existing algorithms
 - Deep Learning (SGD, RMSprop, Adam)
 - Design new algorithms for uncertainty in DL
- Dual perspective of BLR for life-long learning
- Impact: Everything with the same principle



The Bayesian Learning Rule

Mohammad Emtiyaz Khan RIKEN Center for AI Project Tokyo, Japan emtiyaz.khan@riken.jp Håvard Rue CEMSE Division, KAUST Thuwal, Saudi Arabia haavard.rue@kaust.edu.sa

Abstract

We show that many machine-learning algorithms are specific instances of a single algorithm called the *Bayesian learning rule*. The rule, derived from Bayesian principles, yields a wide-range of algorithms from fields such as optimization, deep learning, and graphical models. This includes classical algorithms such as ridge regression, Newton's method, and Kalman filter, as well as modern deep-learning algorithms such as stochastic-gradient descent, RMSprop, and Dropout. The key idea in deriving such algorithms is to approximate the posterior using candidate distributions estimated by using natural gradients. Different candidate distributions result in different algorithms and further approximations to natural gradients give rise to variants of those algorithms. Our work not only unifies, generalizes, and improves existing algorithms, but also helps us design new ones.

Machine Learning from a Bayesian Perspective

Mohammad Emtiyaz Khan RIKEN Center for AI Project Tokyo, Japan emtiyaz.khan@riken.jp

November 8, 2021

Abstract

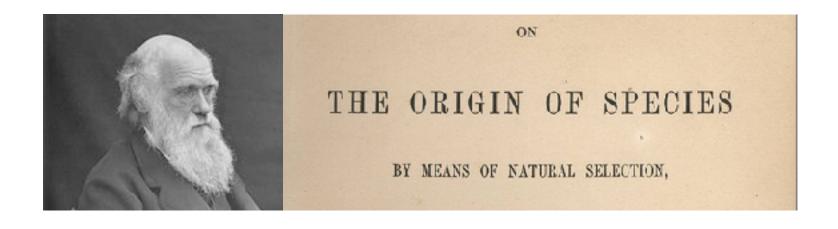
I summarize a Bayesian perspective of machine learning. We view Bayes as an optimization problem whose solutions use the information-geometry of the posterior. Using this perspective, we can show that many machine-learning methods have a (more general) Bayesian side to them. I believe this perspective to be essential for bridging the gap between 'artificial' and 'natural' learning systems.

1 A note about the note

For now, this note deliberately lacks details. One way to read this is to use the accompanying slides. My hope is to add some equations, figures and illustration in the future. Many technical details discussed here can be found in Khan and Rue [2021]

2 Machine learning and Bayes

A main goal of machine-learning is to design AI systems that can learn like us. We humans, and other animals, collect experiences throughout our lives to learn and adapt. Machines currently are extremely bad at this. Majority of successful machine-learning paradigms are the ones that use 'bulk' learning in a 'static' world, where all the information is assumed to be available at once and the world stands still while we learn about it. This is far from the reality of the world we live in, and it is not surprising to see such systems fail. How can we bridge this gap between machines and living-beings? Taking a Bayesian perspective seems to be one way to go, but we argue that this is perhaps the only way forward.



The Origin of Algorithms

A good algorithm must revise its *past* beliefs by using useful *future* information

Principle of Trial-and-Error

Frequentist: Empirical Risk Minimization (ERM) or Maximum Likelihood Principle, etc.

$$\min_{\theta \text{ Loss}} \ell(\mathcal{D}, \theta) = \sum_{i=1}^{N} [y_i - f_{\theta}(x_i)]^2 + \gamma \theta^T \theta$$
 $\max_{\theta \text{ Deep}} \ell(\mathcal{D}, \theta) = \sum_{i=1}^{N} [y_i - f_{\theta}(x_i)]^2 + \gamma \theta^T \theta$
Model Params

Deep Learning Algorithms: $\theta \leftarrow \theta - \rho H_{\theta}^{-1} \nabla_{\theta} \ell(\theta)$

Scales well to large data and complex model, and very good performance in practice.

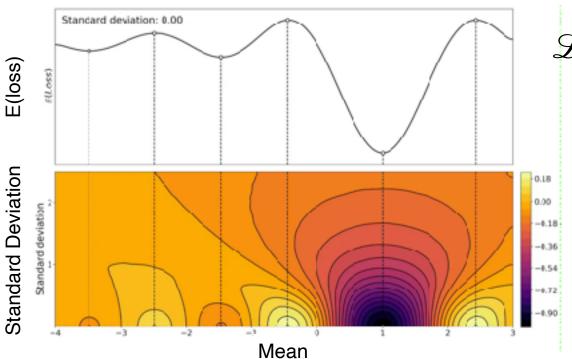
Bayes Objective

$$\min_{\theta} \ \ell(\theta) \quad \text{vs} \quad \min_{q \in \mathcal{Q}} \ \mathbb{E}_{\mathbf{q}(\theta)}[\ell(\theta)] - \mathcal{H}(q) \text{ Entropy }$$
 Generalized-Posterior approx.

- 1. Zellner, A. "Optimal information processing and Bayes's theorem." *The American Statistician* (1988)
- 2. Many other: Bissiri, et al. (2016), Shawe-Taylor and Williamson (1997), Cesa-Bianchi and Lugosi (2006)
- 3. Huszar's blog, Evolution Strategies, Variational Optimisation and Natural ES (2017)
- 4. Smith et al., On the Origin of Implicit Regularization in Stochastic Gradient Descent, ICLR, 2021

Bayes Objective

$$\min_{\theta} \ \ell(\theta) \quad \text{vs} \quad \min_{q \in \mathcal{Q}} \ \mathbb{E}_{\mathbf{q}(\theta)}[\ell(\theta)] - \mathcal{H}(q) \text{ Entropy }$$
 Generalized-Posterior approx.



$$\mathcal{L}(\mu, \sigma) = \mathbb{E}_{\mathcal{N}(\theta | \mu, \sigma^2)}[\ell(\theta)]$$

Instead of the original loss, optimize a different (smoothed) one.

A popular idea of "implicit regularization" in DL [4] now, but also A common idea in many other fields

- 1. Zellner, A. "Optimal information processing and Bayes's theorem." *The American Statistician* (1988)
- 2. Many other: Bissiri, et al. (2016), Shawe-Taylor and Williamson (1997), Cesa-Bianchi and Lugosi (2006)
- 3. Huszar's blog, Evolution Strategies, Variational Optimisation and Natural ES (2017)
- 4. Smith et al., On the Origin of Implicit Regularization in Stochastic Gradient Descent, ICLR, 2021

Bayesian Learning Rule

Unify, generalize, and improve machine-learning algorithms

A 2-step Bayesian Scheme

Step 1: Choose an approximation (mix-exp-family)

Natural parameters Sufficient statistics Expectation parameters
$$q(\theta) \propto \exp\left[\lambda^\top T(\theta)\right] \qquad \qquad \mu := \mathbb{E}_q[T(\theta)]$$

$$\mathcal{N}(\theta|m,S^{-1}) \propto \exp\left[-\frac{1}{2}(\theta-m)^\top S(\theta-m)\right] \\ \propto \exp\left[(Sm)^\top \theta + \operatorname{Tr}\left(-\frac{S}{2}\theta\theta^\top\right)\right]$$

Gaussian distribution
$$q(\theta) := \mathcal{N}(\theta|m, S^{-1})$$

Natural parameters $\lambda := \{Sm, -S/2\}$
Expectation parameters $\mu := \{\mathbb{E}_q(\theta), \mathbb{E}_q(\theta\theta^\top)\}$

A 2-step Bayesian Scheme

Step 2:
$$\min_{\theta} \ \ell(\theta)$$
 vs $\min_{q \in \mathcal{Q}} \mathbb{E}_{q(\theta)}[\ell(\theta)] - \mathcal{H}(q)$ Exponential-family Approx.

Deep Learning algo:
$$\theta \leftarrow \theta - \rho H_{\theta}^{-1} \nabla_{\theta} \ell(\theta)$$

Bayes learning rule: $\lambda \leftarrow \lambda - \rho \nabla_{\mu} \left(\mathbb{E}_q[\ell(\theta)] - \mathcal{H}(q) \right)$

1 Natural Gradient

Natural and Expectation parameters of an exponential family distribution q (natural-gradient descent & mirror descent)

By changing Q, we can recover DL algorithms (and more)

Bayesian learning rule: $\lambda \leftarrow \lambda - \rho \nabla_{\mu} \left(\mathbb{E}_q[\ell(\theta)] - \mathcal{H}(q) \right)$

Learning Algorithm	Posterior Approx.	Natural-Gradient Approx.	Sec.		
Optimization Algorithms					
Gradient Descent	Gaussian (fixed cov.)	Delta method	1.3		
Newton's method	Gaussian		1.3		
Multimodal optimization (New)	Mixture of Gaussians	"	3.2		
	Deep-Learning Algor	rithms			
Stochastic Gradient Descent	Gaussian (fixed cov.)	Delta method, stochastic approx.	4.1		
RMSprop/Adam	Gaussian (diagonal cov.)	Delta method, stochastic approx., Hessian approx., square-root scal- ing, slow-moving scale vectors	4.2		
Dropout	Mixture of Gaussians	Delta method, stochastic approx., responsibility approx.	4.3		
STE	Bernoulli	Delta method, stochastic approx.	4.5		
Online Gauss-Newton (OGN) $_{(New)}$	Gaussian (diagonal cov.)	Gauss-Newton Hessian approx. in Adam & no square-root scaling	4.4		
Variational OGN (New)	"	Remove delta method from OGN	4.4		
$BayesBiNN_{\rm \ (New)}$	Bernoulli	Remove delta method from STE	4.5		
Appro	oximate Bayesian Infere	nce Algorithms			
Conjugate Bayes	Exp-family	Set learning rate $\rho_t = 1$	5.1		
Laplace's method	Gaussian	Delta method	4.4		
Expectation-Maximization	Exp- $Family + Gaussian$	Delta method for the parameters	5.2		
Stochastic VI (SVI)	Exp-family (mean-field)	Stochastic approx., local $\rho_t = 1$	5.3		
VMP	"	$ \rho_t = 1 \text{ for all nodes} $	5.3		
Non-Conjugate VMP	"	"	5.3		
Non-Conjugate VI (New)	Mixture of Exp-family	None	5.4		

- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." ICML (2018).
- 2. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).

Gradient Descent from Bayes

Gradient descent: $\theta \leftarrow \theta - \rho \nabla_{\theta} \ell(\theta)$

Bayes Learn Rule: $m \leftarrow m - \rho \nabla_m \ell(m)$

"Global" to "local" (the delta method) $\mathbb{E}_q[\ell(heta)] pprox \ell(m)$

$$m \leftarrow m - \rho \nabla_{\mathbf{m}} \mathbb{E}_{q}[\ell(\theta)]$$

$$\mathbb{E}_{q}[\ell(\theta)] \approx \ell(m) \qquad \lambda \leftarrow \lambda - \rho \nabla_{\mu} \left(\mathbb{E}_{q}[\ell(\theta)] - \mathcal{H}(q) \right)$$

Derived by choosing Gaussian with fixed covariance

Gaussian distribution $q(\theta) := \mathcal{N}(m, 1)$

Natural parameters $\lambda := m$

Expectation parameters $\mu := \mathbb{E}_q[\theta] = m$

Entropy $\mathcal{H}(q) := \log(2\pi)/2$

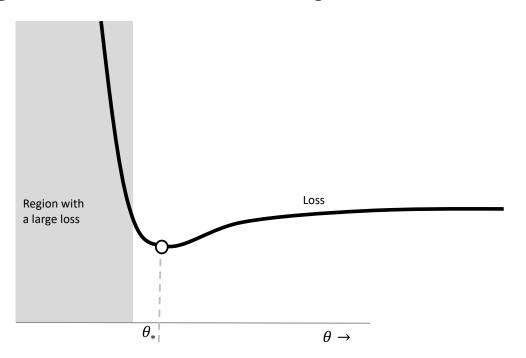
Bayes vs Non-Bayes

GD:
$$\theta \leftarrow \theta - \rho \nabla_{\theta} \ell(\theta) \implies \nabla_{\theta} \ell(\theta_*) = 0$$

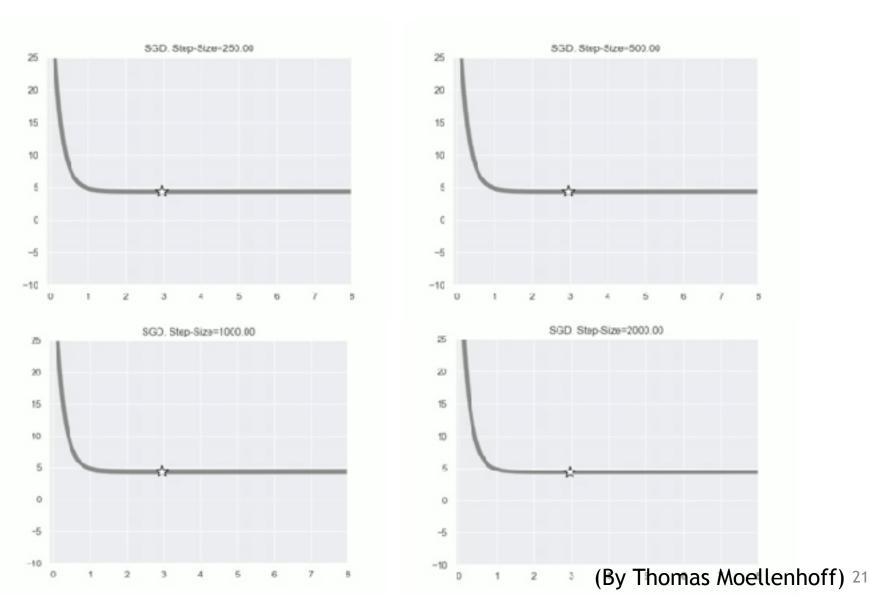
BLR:
$$m \leftarrow m - \rho \nabla_{\mathbf{m}} \mathbb{E}_q[\ell(\theta)]$$

$$\Longrightarrow \nabla_m \mathbb{E}_{q_*}[\ell(\theta)] = 0 \qquad \Longrightarrow \mathbb{E}_{q_*}[\nabla_{\theta} \ell(\theta)] = 0$$

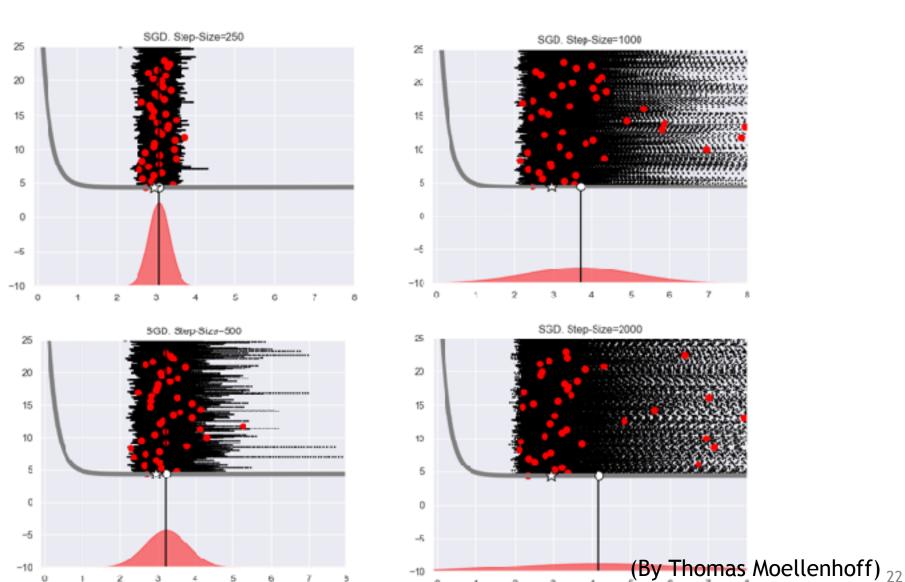
Bayesian solution injects "noise" which has a similar regularization effect to noise in Stochastic GD. It prefers "flatter" directions.



SGD: Implicit Regularization

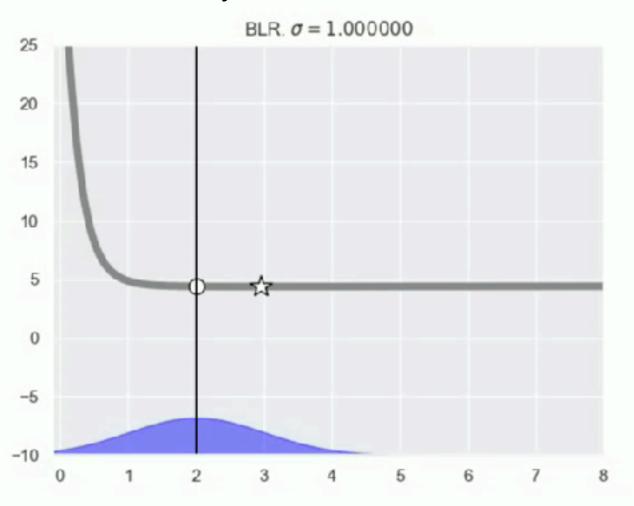


SGD: Implicit Regularization

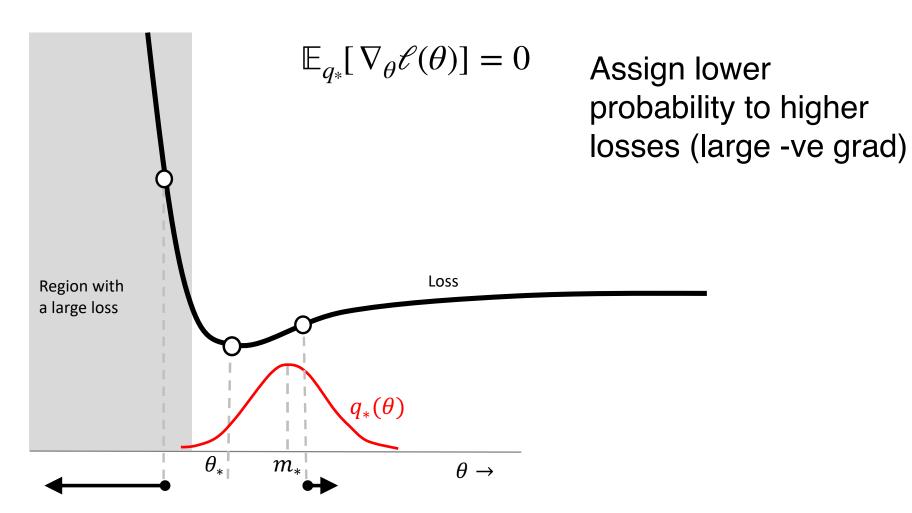


Bayes: Implicit Regularization

Estimating Gaussian posteriors where the variance is fixed, and only the mean is estimated

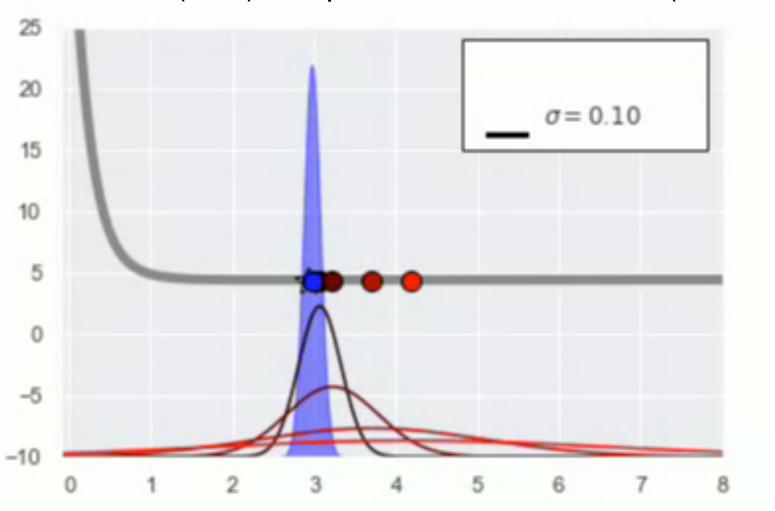


Bayes: Implicit Regularization



Bayes: Implicit Regularization

Bayes solutions (blue) compared to SGD solutions (red lines)



Deriving Learning-Algorithms from the Bayesian Learning Rule

Posterior Approximation \longleftrightarrow Learning-Algorithm



Newton's Method from Bayes

Newton's method: $\theta \leftarrow \theta - H_{\theta}^{-1} \left[\nabla_{\theta} \ell(\theta) \right]$

$$Sm \leftarrow (1 - \rho)Sm - \rho \nabla_{\mathbb{E}_{q}(\theta)} \mathbb{E}_{q}[\ell(\theta)]$$

$$-\frac{1}{2}S \leftarrow (1(1 - \rho)S) \frac{1}{2}Sp_{2}\nabla_{\rho}\nabla_{q}(\theta)\mathbb{E}_{q}[\ell(\theta)]$$

$$\lambda \leftarrow (1-\rho) \text{ for } (\mathbb{E}_{q} \mathbb{V}(\theta))_{q} \mathbb{H}(q)) \qquad \boxed{-\nabla_{\mu}\mathcal{H}(q) = \lambda}$$

Derived by choosing a multivariate Gaussian

 $\begin{array}{ll} \text{Gaussian distribution} & q(\theta) := \mathcal{N}(\theta|m,S^{-1}) \\ \text{Natural parameters} & \lambda := \{Sm,-S/2\} \\ \text{Expectation parameters} & \mu := \{\mathbb{E}_q(\theta),\mathbb{E}_q(\theta\theta^\top)\} \end{array}$

Newton's Method from Bayes

Newton's method: $\theta \leftarrow \theta - H_{\theta}^{-1} \left[\nabla_{\theta} \ell(\theta) \right]$

Set
$$\rho$$
 =1 to get $m \leftarrow m - H_m^{-1}[\nabla_m \ell(m)]$

$$m \leftarrow m - \rho S^{-1} \nabla_m \ell(m)$$
$$S \leftarrow (1 - \rho)S + \rho H_m$$

Delta Method $\mathbb{E}_q[\ell(\theta)] \approx \ell(m)$

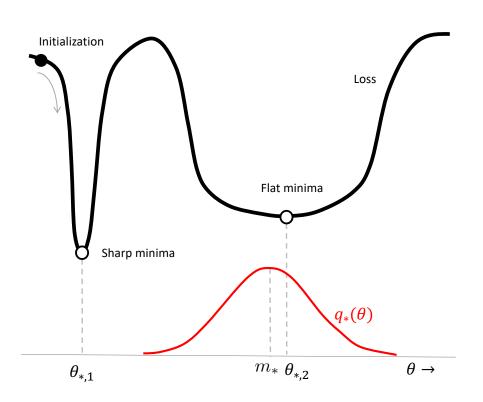
Express in terms of gradient and Hessian of loss:

$$\nabla_{\mathbb{E}_q(\theta)} \mathbb{E}_q[\ell(\theta)] = \mathbb{E}_q[\nabla_{\theta} \ell(\theta)] - 2\mathbb{E}_q[H_{\theta}] m$$
$$\nabla_{\mathbb{E}_q(\theta\theta^{\top})} \mathbb{E}_q[\ell(\theta)] = \mathbb{E}_q[H_{\theta}]$$

$$Sm \leftarrow (1 - \rho)Sm - \rho \nabla_{\mathbb{E}_{q}(\theta)} \mathbb{E}_{q}[\ell(\theta)]$$
$$S \leftarrow (1 - \rho)S - \rho 2 \nabla_{\mathbb{E}_{q}(\theta\theta^{\top})} \mathbb{E}_{q}[\ell(\theta)]$$

Bayes leads to robust solutions

Avoiding sharp minima



Bayesian learning rule: $\lambda \leftarrow \lambda - \rho \nabla_{\mu} \left(\mathbb{E}_q[\ell(\theta)] - \mathcal{H}(q) \right)$

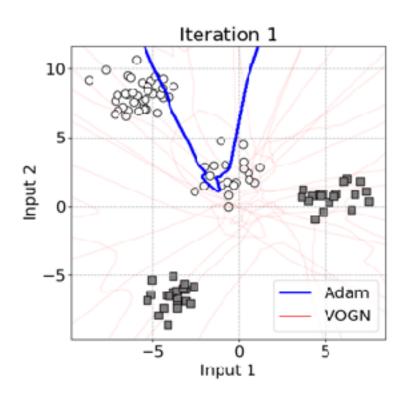
Learning Algorithm	Posterior Approx.	Natural-Gradient Approx.	Sec.	
Optimization Algorithms				
Gradient Descent	Gaussian (fixed cov.)	Delta method	1.3	
Newton's method	Gaussian		1.3	
Multimodal optimization (New)	Mixture of Gaussians	"	3.2	
Deep-Learning Algorithms				
Stochastic Gradient Descent	Gaussian (fixed cov.)	Delta method, stochastic approx.	4.1	
RMSprop/Adam	Gaussian (diagonal cov.)	Delta method, stochastic approx., Hessian approx., square-root scal- ing, slow-moving scale vectors	4.2	
Dropout	Mixture of Gaussians	Delta method, stochastic approx., responsibility approx.	4.3	
STE	Bernoulli	Delta method, stochastic approx.	4.5	
Online Gauss-Newton (OGN) (New)	Gaussian (diagonal cov.)	Gauss-Newton Hessian approx. in Adam & no square-root scaling	4.4	
Variational OGN (New)	"	Remove delta method from OGN	4.4	
BayesBiNN (New)	Bernoulli	Remove delta method from STE	4.5	
Approximate Bayesian Inference Algorithms				
Conjugate Bayes	Exp-family	Set learning rate $\rho_t = 1$	5.1	
Laplace's method	Gaussian	Delta method	4.4	
Expectation-Maximization	Exp- $Family + Gaussian$	Delta method for the parameters	5.2	
Stochastic VI (SVI)	Exp-family (mean-field)	Stochastic approx., local $\rho_t = 1$	5.3	
VMP	"	$ \rho_t = 1 $ for all nodes	5.3	
Non-Conjugate VMP	u	"	5.3	
Non-Conjugate VI (New)	Mixture of Exp-family	None	5.4	

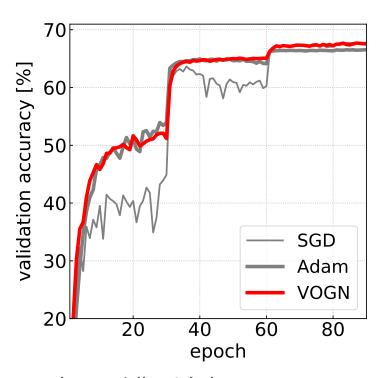
We can compute uncertainty using a variant of Adam.

- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
- 2. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).

Uncertainty of Deep Nets

VOGN: A modification of Adam but match the performance on ImageNet





Code available at https://github.com/team-approx-bayes/dl-with-bayes

- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
- 2. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).

RMSprop/Adam from Bayes

RMSprop

BLR for Gaussian approx

$$s \leftarrow (1 - \rho)s + \rho[\hat{\nabla}\ell(\theta)]^2$$
$$\theta \leftarrow \theta - \alpha(\sqrt{s} + \delta)^{-1}\hat{\nabla}\ell(\theta)$$

$$S \leftarrow (1 - \rho)S + \rho(\mathbf{H}_{\theta})$$
$$m \leftarrow m - \alpha S^{-1} \nabla_{\theta} \ell(\theta)$$

To get RMSprop, make the following choices

- Restrict covariance to be diagonal
- Replace Hessian by square of gradients
- Add square root for scaling vector

For Adam, use a Heavy-ball term with KL divergence as momentum (Appendix E in [1])

Variational Online Newton Methods

RMSprop

Variational Online Gauss-Newton

$$g \leftarrow \hat{\nabla}\ell(\theta)$$

$$s \leftarrow (1 - \rho)s + \rho g^{2}$$

$$\theta \leftarrow \theta - \alpha(\sqrt{s} + \delta)^{-1}g$$

$$g \leftarrow \hat{\nabla}\ell(\theta)$$
, where $\theta \sim \mathcal{N}(m, \sigma^2)$
 $s \leftarrow (1 - \rho)s + \rho(\Sigma_i g_i^2)$
 $m \leftarrow m - \alpha(s + \gamma)^{-1} \nabla_{\theta}\ell(\theta)$
 $\sigma^2 \leftarrow (s + \gamma)^{-1}$

Available at https://github.com/team-approx-bayes/dl-with-bayes

The BLR variant from [3] led to the winning solution for the NeurIPS 2021 challenge for "approximate inference in deep learning". Watch Thomas Moellenhoff's talk at https://www.youtube.com/watch?v=LQInIN5EU7E.



- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." *ICML* (2018).
- 2. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).
- 3. Lin et al. "Handling the positive-definite constraints in the BLR." ICML (2020).

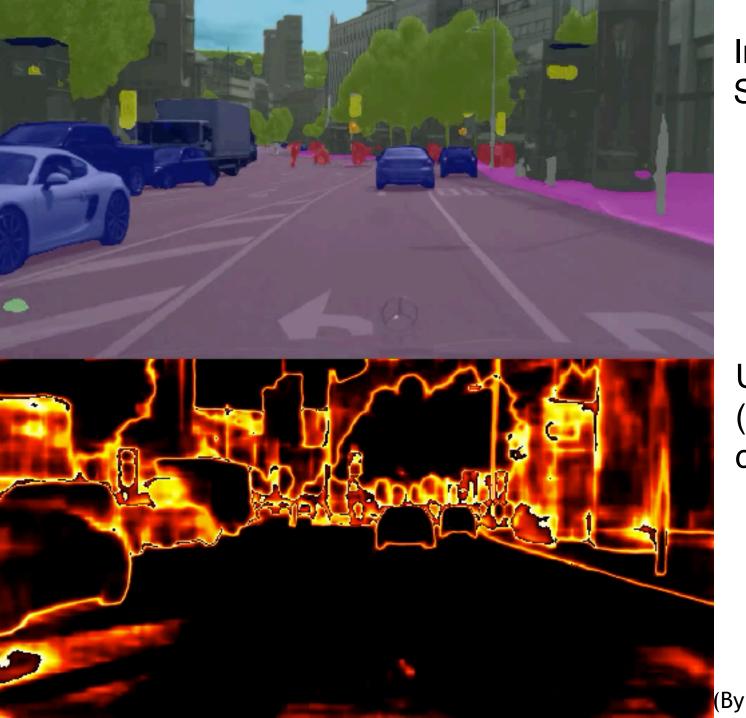
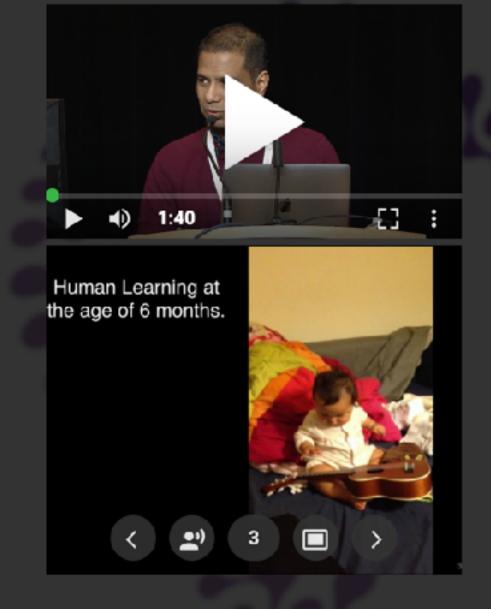


Image Segmentation

Uncertainty (entropy of class probs)

(By Roman Bachmann)34



Deep Learning with **Bayesian Principles** NEURAL INFORMATION PRICESSANG STREETS

by Mohammad Emtiyaz Khan · Dec 9, 2019

NeurlPS 2019 **Tutorial**

#NeurIPS 2019



Views 151 807

Presentations 263

Followers 200



From System 1 Deep Learning to System 2 Deep Learning

by Yoshua Benglo

17,953 views - Dec 11, 2019.

8,084 views · Dec 9, 2019



NeurIPS Workshop on Machine Learning for Creativity and Design...

by Aaron Hertzmann, Adam Roberts, ...

9,654 views : Dec 14, 2019





Efficient Processing of Deep Neural Network: from Algorithms to...

by Wylenne Sze

7.163 views : Dec 9, 2019.

Past and New Work

Natural Gradient Variational Inference

- 1. Khan and Lin. "Conjugate-computation variational inference: Converting variational inference in non-conjugate models to inferences in conjugate models." Alstats (2017).
- 2. Khan and Nielsen. "Fast yet simple natural-gradient descent for variational inference in complex models." (2018) ISITA.



3. Lin et al. "Fast and Simple Natural-Gradient Variational Inference with Mixture of Exponential-family Approximations," ICML (2019).



- 4. Lin et al. "Handling the Positive-Definite Constraint in the Bayesian Learning Rule", ICML (2020)
- 5. Lin et al. "Tractable structured natural gradient descent using local parameterizations", ICML, (2021)
- Gaussian approx ←→ Newton-variants



Wu Lin (UBC)



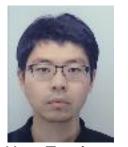
Mark Schmidt (UBC)



Frank Nielsen (Sony)

Gaussian Approximation and DL

- 1. Khan, et al. "Fast and scalable Bayesian deep learning by weight-perturbation in Adam." ICML (2018).
- 2. Mishkin et al. "SLANG: Fast Structured Covariance Approximations for Bayesian Deep Learning with Natural Gradient" NeurIPS (2018).
- 3. Osawa et al. "Practical Deep Learning with Bayesian Principles." NeurIPS (2019).



Voot Tangkaratt (Postdoc, RIKEN-AIP)







Yarin Gal (UOxford)



Akash Srivastava (UEdinburgh)



Kazuki Osawa (Tokyo Tech)



Rio Yokota (Tokyo Tech)



Anirudh Jain (Intern from IIT-ISM, India)



Runa Eschenhagen (Intern from U Osnabruck)



Siddharth Swaroop (UCambridge)



Rich Turner (UCambridge)

Extensions

- Binary Neural Networks (Bernoulli approx)
 - 1. Meng, et al. "Training Binary Neural Networks using the Bayesian Learning Rule." *ICML* (2020).
- Gaussian Process
 - 2. Chang et al. "Fast Variational Learning in State-Space GP Models", MLSP (2020)
 - For sparse GPs, BLR is a generalization of [1]



Roman Bachmann (Intern from EPFL)



Xiangming Meng (RIKEN-AIP)



Paul Chang (Aalto University)



W. J. Wilkinson (Aalto University)



Arno Solin (Aalto University)

How to design AI that learn like us?

- Three questions
 - Q1: What do we know? (model)
 - Q2: What do we not know? (uncertainty)
 - Q3: What do we need to know? (action & exploration)
- Posterior approximation is the key
 - (Q1) Models == representation of the world
 - (Q2) Posterior approximations == representation of the model
 - (Q3) Use posterior approximations for knowledge representation, transfer, and collection.

Approximate Bayesian Inference Team



Emtiyax Khan Team Leader



Pierre Alquier Research Scientist



Gian Maria Marconi Postdoc



Thomas Möllenhoff Postdoc

https://team-approx-bayes.github.io/

We have many open positions! Come, join us.



Lu Xu Postdoc



Jooyeon Kim Postdac



Wu Lin PhD Student University of British Columbia



David Tomàs Cuesta Rotation Student, Okinawa Institute of Science and Technology



Dharmesh Tallor Remote Collaborator University of Amsterdam



Erik Daxberger Remote Collaborator University of Cambridge



Tojo Rakotoaritina Rotation Student, Okinawa Institute of Science and Technology



Peter Nicki Research Assistant



Happy Buzaaba Part-time Student University of Tsukuba



Siddharth Swaroop Remote Collaborator University of

Cambridge



Alexandre Piché
Remote
Collaborator
MILA



Paul Chang Remote Collaborator Aalto University