Variational Methods for Discrete-Data Latent Gaussian Models

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The Big Picture

• Joint density models for data with mixed data types
• Bayesian models - principled and robust approach
• Algorithms that are not only accurate and fast, but are also easy to tune, implement, and intuitive (speed-accuracy tradeoffs)
Sources of Discrete Data

User rating data

Survey/voting data and blogs for sentiment analysis

Health data
tag correlation.

Consumer choice data

Sports/game data
**Motivation: Recommendation system**

Movie rating dataset - Missing values - Different types of data

<table>
<thead>
<tr>
<th></th>
<th>User1</th>
<th>User2</th>
<th>User3</th>
<th>User4</th>
<th>User5</th>
<th>User6</th>
<th>....</th>
</tr>
</thead>
<tbody>
<tr>
<td>Movie1</td>
<td>9</td>
<td>2</td>
<td>3</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Movie2</td>
<td>8</td>
<td></td>
<td>8</td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Movie3</td>
<td></td>
<td>2</td>
<td>8</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Movie4</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Movie5</td>
<td>2</td>
<td>7</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Movie6</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Missing Ratings

From Wikipedia on Netflix-prize dataset

“The training set is such that the average user rated over 200 movies, and the average movie was rated by over 5000 users. But there is wide variance in the data—some movies in the training set have as few as 3 ratings, while one user rated over 17,000 movies.”

Movielens Dataset
Sources of Discrete Data

- User rating data
- Survey/voting data and blogs for sentiment analysis
- Health data
- Consumer choice data
- Sports/game data
- Tag correlation.
What we need!

For these datasets, we need a method of analysis which

- Handles missing values efficiently
- Makes efficient use of the data by weighting “reliable” data vectors more than the “unreliable” ones
- Makes efficient use of the data by “fusing” different types of data efficiently (binary, ordinal, categorical, count, text)
Factor Model

\[ \begin{align*}
\text{ExpFamily:} & \quad \min_W \sum_{n=1}^N \min_{z_n} - \log p(y_n | z_n, W) \\
\text{Gaussian:} & \quad \min_W \sum_{n=1}^N \min_{z_n} \| y_n - W z_n \|^2_2
\end{align*} \]

Bayesian Learning

\[
\max_{\theta} \sum_{n=1}^{N} \log \int p(y_n | z_n, W) \mathcal{N}(z_n | \mu, \Sigma) \, dz_n
\]

\[
\max_{\theta} \sum_{n=1}^{N} \log \int \mathcal{N}(z_n | \mu, \Sigma) \, dx \, dz
\]

\[
\geq \max_{\theta} \sum_{n=1}^{N} \max_{\psi_n} \mathcal{L}_n(\theta, \psi_n)
\]

This talk: Lower bound maximization
Variational Methods

\[
\max_\theta \sum_{n=1}^N \log \int p(y_n | z_n, W) \mathcal{N}(z_n | \mu, \Sigma) dz_n \geq \max_\theta \sum_{n=1}^N \max_{\psi_n} \mathcal{L}_n(\theta, \psi_n)
\]

- Design tractable bounds to reduce approximation error
- Efficient optimization since lower bounds are concave: good convergence rates and easy convergence diagnostics
- Efficient expectation-maximization (EM) algorithms for parameter leaning
- Comparable performance to MCMC, but much faster
- Algorithms with a wide range of speed-accuracy trade-offs
Outline

- Latent Gaussian models
- Bounds for binary data
- Bounds for categorical data
- Results
- Future work and conclusions
Outline

- Latent Gaussian models
  - Definition and examples
  - Problem with parameter learning
- Bounds for binary data
- Bounds for categorical data
- Results
- Future work and conclusions
Latent Gaussian Model (LGM)

\[ p(y_{dn} | \eta_{dn}) \]

\[ \eta_{dn} = W_d z_n \]

\[ p(z_n | \theta) = \mathcal{N}(z_n | \mu, \Sigma) \]
### Likelihood Examples

<table>
<thead>
<tr>
<th>Data type</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real</td>
<td>Gaussian</td>
</tr>
<tr>
<td>Count</td>
<td>Poisson</td>
</tr>
<tr>
<td>Binary</td>
<td>Bernoulli-Logit</td>
</tr>
<tr>
<td>Categorical</td>
<td>Multinomial-Logit</td>
</tr>
<tr>
<td>Ordinal</td>
<td>Proportional-odds</td>
</tr>
</tbody>
</table>

For a binary variable $y$:

$$p(y = 1 | \eta) = \frac{e^{\eta}}{1 + e^{\eta}}$$

For a categorical variable $y$:

$$p(y = k | \eta) = \frac{e^{\eta_k}}{\sum_{j=1}^{K} e^{\eta_j}}$$
Parameter Estimation

\[
\max_{\theta} \sum_{n=1}^{N} \log \int \prod_{d=1}^{D} p(y_{dn} | z, \theta) N(z | \mu, \Sigma) dz
\]

Bernoulli-logit

\[
\log \int \prod_{d=1}^{D} \text{Bernoulli-logit} \times dZ
\]
Jensen’s Lower Bound

\[ \mathcal{L}(\theta | y) = \log \int \prod_{d=1}^{D} p(y_d | z, \theta) \mathcal{N}(z | \mu, \Sigma) dz \]

\[ = \log \int \frac{\prod_{d=1}^{D} p(y_d | z, \theta) \mathcal{N}(z | \mu, \Sigma)}{\mathcal{N}(z | m, V)} \mathcal{N}(z | m, V) dz \]

\[ \mathcal{L}(\theta | y) \geq \max_{m, V} \sum_{d=1}^{D} \int [\log p(y_d | z, \theta)] \mathcal{N}(z | m, V) dz \]

\[ - KL [ \mathcal{N}(m, V) || \mathcal{N}(\mu, \Sigma)] \]
Variational Methods for Discrete-Data Latent Gaussian Models

Variational Lower Bound

\[
\max_{\theta} \sum_{n=1}^{N} \max_{m_n, V_n} \sum_{d=1}^{D} \int \log p(y_{dn} | \eta_{dn}) \mathcal{N}(z | m_n, V_n) dz \\
- KL[\mathcal{N}(m_n, V_n) || \mathcal{N}(\mu, \Sigma)]
\]

- Generalized EM algorithm
- E-step involves minimizing convex function
- Early stopping in E-step
- (Almost) no tuning parameters
- Easy convergence diagnostics
Outline

• Latent Gaussian models
• Bounds for binary data
  • Bernoulli-logistic likelihood
  • The Bohning bound *(Khan, Marlin, Bouchard, Murphy, NIPS 2010)*
  • Piecewise bounds *(Marlin, Khan, Murphy, ICML 2011)*
• Bounds for categorical data
• Results
• Future work and conclusions
Bernoulli-Logit Likelihood

\[ p(y = 1 | \eta) = \frac{e^\eta}{1 + e^\eta} \]

\[ \log p(y = 1 | \eta) = \eta - \log(1 + e^\eta) \]

\[ \mathcal{L}(\theta | y) \geq \max_{m, V} \sum_{d=1}^{D} \int \left[ \log p(y_d | z, \theta) \right] \mathcal{N}(z | m, V) dz - KL [\mathcal{N}(m, V) || \mathcal{N}(\mu, \Sigma)] \]

\[ = \max_{m, V} \sum_{d=1}^{D} \int \left[ -\log(1 + e^\eta) \right] \mathcal{N}(\tilde{m}_d, \tilde{\nu}_d) d\eta + \text{some other tractable terms in } m \text{ and } V \]
Local Variational Bounds

\[
\max_{m,V} \sum_{d=1}^{D} \int \left[ -\log(1 + e^\eta) \right] \mathcal{N}(\tilde{m}_d, \tilde{v}_d) d\eta + \text{some other tractable terms in } m \text{ and } V
\]

- Bohning’s bound \((\text{Khan, Marlin, Bouchard, Murphy 2010})\)
- Jaakola’s bound \((\text{Jaakkola and Jordan 1996})\)
- Piecewise quadratic bounds \((\text{Marlin, Khan, Murphy 2011})\)
Bohning Bound is Faster

Variational Methods for Discrete-Data Latent Gaussian Models

For $n = 1:N$

$$V_n = (W^T A_n W + I)^{-1}$$

$$m_n = \ldots$$

end

Jaakkola Bound

Bohning Bound

$O(L^3 ND)$

$O(L^2 ND)$

$V = (W^T AW + I)^{-1}$

For $n = 1:N$

$$m_n = \ldots$$

end
Piecewise bounds are more accurate

Bohning

Jaakkola

Piecewise

\[ Q_1(x) \]

\[ Q_2(x) \]

\[ Q_3(x) \]

\[ t_0 = -\infty \]

\[ t_1 \]

\[ t_2 \]

\[ t_3 = \infty \]
Details of Piecewise bounds

- Find cut points and parameters of each piece by minimizing maximum error
- Linear pieces (Hsiung, Kim and Boyd, 2008)
- Quadratic Pieces (Nelder-Mead method)
- Fixed Piecewise Bounds!
- Increase accuracy by increasing the number of pieces
Outline

- Latent Gaussian models
- Bounds for binary data
- Bounds for categorical data
  - Multinomial-logistic likelihood and local variational bounds
  - Stick-breaking likelihood (Khan, Mohamed, Marlin, Murphy, AI-Stats 2012)
- Results
- Future work and conclusions
Multinomial-Logit Likelihood

\[
p(y = k|\eta) = \frac{e^{\eta_k}}{\sum_{j=1}^{K} e^{\eta_j}}
\]

\[
\log p(y = k|\eta) = \eta_k - \log \sum_{j=1}^{K} e^{\eta_j}
\]

\[
\mathcal{L}(\theta|y) \geq \max_{m, V} \sum_{d=1}^{D} \int [\log p(y_d|z, \theta)] \mathcal{N}(z|m, V) dz
\]

\[
- KL [\mathcal{N}(m, V)||\mathcal{N}(\mu, \Sigma)]
\]
Local Variational bounds

- The Bohning bound
  - Fast and closed form updates
- The log bound \cite{Blei2006}
  - More accurate than the Bohning bound, but slower
- The product of sigmoid bound \cite{Bouchard2007}
Variational Methods for Discrete-Data Latent Gaussian Models

Stick-Breaking Likelihood

\[ p(y = 1|\eta) = \frac{e^{\eta_1}}{1 + e^{\eta_1}} \]

\[ p(y = 2|\eta) = \left(1 - \frac{e^{\eta_1}}{1 + e^{\eta_1}}\right) \frac{e^{\eta_2}}{1 + e^{\eta_2}} \]

\[ p(y = 3|\eta) = \left(1 - \frac{e^{\eta_1}}{1 + e^{\eta_1}}\right) \left(1 - \frac{e^{\eta_2}}{1 + e^{\eta_2}}\right) \frac{e^{\eta_3}}{1 + e^{\eta_3}} \]

\[ \vdots \]

\[ p(y = K|\eta) = \prod_{j=1}^{K-1} \left(1 - \frac{e^{\eta_j}}{1 + e^{\eta_j}}\right) \]

\[ \log p(y = k|\eta) = \eta_k - \sum_{j=1}^{K-1} I(j \leq k) \log (1 + e^{\eta_j}) \]
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Speed Accuracy Trade-offs

Binary FA : UCI voting dataset (D=15, N=435)
Comparison with EP

Binary Gaussian Process: Ionosphere dataset (D=200)

\[ \Sigma_{ij} = \sigma \exp[-||x_i-x_j||^2/s] \]
EP vs PW: Posterior Distribution

(Neg) KL-Lower Bound to MargLik

Approximation To MargLik

Pred Error
Comparison with EP

- Both methods give very similar results for GPs
- Our approach can be easily extended to factor models
- Variational EM objective function is well-defined and can be obtained by solving minimization of convex functions
- Numerically stable
MultiClass Gaussian Process

Glass dataset (D=143, K=6)

MCMC  Bohning  Log  VB-probit  Stick-PW

NegLogLik

\[
\log(\sigma)
\]

Prediction Error

\[
\log(\sigma)
\]
Categorical Factor Analysis

Glass dataset (D = 10, N = 958, sum of K = 29)
Outline

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Future Work

- Large-scale collaborative filtering
- Use convexity to design approximate gradient methods
- Sparse Gaussian Posterior Distribution
- Tuning HMC using Bayesian optimization methods
- Latent Sparse-factor model
- Conditional models (e.g. to model for tag-image correlation)
Conclusions

- Variational methods show comparable performance with existing approaches
- The main sources of errors is the bounding error
- Design of piecewise bounds to control these errors
- A good control over speed-accuracy trade-offs can be obtained
- Variational lower bounds can be optimized efficiently
  - Use of convex optimization methods to get fast convergence rates and easy convergence diagnostics
  - Design of efficient expectation-maximization (EM) algorithms
Collaborators

Kevin Murphy
UBC

Benjamin Marlin
U. Mass-Amherst

Guillaume Bouchard
XRCE, France

Shakir Mohamed
U. Cambridge, now at UBC
Thank You