Updating Inverse of a Matrix When a Column is Added/Removed

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Abstract

Given a matrix X with inverse $(X^T X)^{-1}$, we describe an update rule to compute inverses when a column is added and removed.

1 Matrix-Inversion Lemma

Given matrix A,U, C and V of right sizes, matrix-inversion-lemma gives the following expression for the inverse

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \tag{1}$$

This can be used to compute inverse of update of the following form:

$$(A + \mathbf{u}\mathbf{v}^T)^{-1} = A^{-1} - cA^{-1}\mathbf{u}\mathbf{v}^T A^{-1} \tag{2}$$

where $c = 1/(1 + \mathbf{u}^T A^{-1}\mathbf{v})$. See [1] for various application of matrix-inversion lemma.

2 Inverting Partitioned Matrix

Inverse of a partitioned matrix can be written as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} F_{11}^{-1} & -F_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}F_{11}^{-1} & F_{22}^{-1} \end{bmatrix}^{-1} \tag{3}$$

where,

$$F_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21} \tag{4}$$
$$F_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12} \tag{5}$$

We can use matrix-inversion lemma to find the F_{11}^{-1} and F_{22}^{-1}:

$$F_{11}^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}F_{22}^{-1}A_{21}A_{11}^{-1} \tag{6}$$
$$F_{22}^{-1} = A_{22}^{-1} + A_{22}^{-1}A_{21}F_{11}^{-1}A_{12}A22^{-1} \tag{7}$$

This gives us several other ways of writing the inverse of above partitioned matrix:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} F_{11}^{-1} & -A_{11}^{-1}A_{12}F_{22}^{-1} \\ -F_{22}^{-1}A_{21}A_{11}^{-1} & F_{22}^{-1} \end{bmatrix}^{-1} \tag{8}$$

$$= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}F_{22}^{-1}A_{21}A_{11}^{-1} & -F_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}F_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}F_{11}^{-1}A_{12}A22^{-1} \end{bmatrix}^{-1} \tag{9}$$

These formula are taken from [2].

3 Addition or Deletion of a Column

Let X be a matrix of size $n \times p$. Let us say we have already computed the following inverse: $B = (X^T X)^{-1}$. Now if we add a column $\mathbf{v}$ to X so that $\tilde{X} = [X \quad \mathbf{v}]$, then we want to computer $\tilde{B} = (\tilde{X}^T \tilde{X})^{-1}$ given $B = (X^T X)^{-1}$. We have,

$$\begin{aligned}
\tilde{B}^{-1} &= \begin{bmatrix} X^T \\ \mathbf{v}^T \end{bmatrix} \begin{bmatrix} X & \mathbf{v} \end{bmatrix} && (10) \\
&= \begin{bmatrix} X^T X & X^T \mathbf{v} \\ \mathbf{v}^T X & \mathbf{v}^T \mathbf{v} \end{bmatrix} && (11)
\end{aligned}$$

Using inverse of a partitioned matrix as in Eq. (3),

$$\begin{aligned}
\tilde{B} &= \begin{bmatrix} X^T X & X^T \mathbf{v} \\ \mathbf{v}^T X & \mathbf{v}^T \mathbf{v} \end{bmatrix}^{-1} && (12) \\
&= \begin{bmatrix} F_{11}^{-1} & -dBX^T \mathbf{v} \\ -d\mathbf{v}^T X B^T & d \end{bmatrix}^{-1} && (13)
\end{aligned}$$

where

$$\begin{aligned}
d &= \frac{1}{\mathbf{v}^T \mathbf{v} - \mathbf{v}^T X B X^T \mathbf{v}} && (14) \\
F_{11}^{-1} &= B + dBX^T \mathbf{v}\mathbf{v}^T X B^T && (15)
\end{aligned}$$

If you are not adding a column as the last column of the matrix, but within the matrix somewhere, then you need to permute the result at the end. Here is the pseudo-code based on this:

Algorithm 1: One-Rank update to $(X^T X)^{-1}$, when a column $\mathbf{v}$ is added to X at position j

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$\mathbf{u}_1 \leftarrow X^T \mathbf{v}$
$\mathbf{u}_2 \leftarrow B\mathbf{u}_1$
$\mathbf{u}_3 \leftarrow d\mathbf{u}_2$
$F_{11}^{-1} \leftarrow B + d\mathbf{u}_2^T \mathbf{u}_2$
$d \leftarrow 1/(\mathbf{v}^T \mathbf{v} - \mathbf{u}_1^T \mathbf{u}_2)$
$\tilde{B} \leftarrow \begin{bmatrix} F_{11}^{-1} & -\mathbf{u}_3 \\ -\mathbf{u}_3^T & d \end{bmatrix}$
Permute column j and row j of $\tilde{B}$ to last column and last row

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Now consider the case when we need to remove a column from matrix X. We can find an update by interchanging $\tilde{B}$ and B. We do not discuss it as it is straightforward. Here is the pseudo-code for inverse update:

Algorithm 2: One-Rank update when a column is removed

Permute column j and row j of B to last column and last row
$F_{11}^{-1} \leftarrow B(1 : p - 1.1 : p - 1)$
$d \leftarrow B(p, p)$
$\mathbf{u}_3 \leftarrow -B(1 : p - 1, p)$
$\mathbf{u}_2 \leftarrow \mathbf{u}_3 / d$
$\tilde{B}^{-1} \leftarrow F_{11}^{-1} - d\mathbf{u}_2 \mathbf{u}_2^T$

These two algorithms are implemented in `OneColInv.m`.

References

[1] Hager, W.W.,Updating the Inverse of a Matrix,SIAM Review, vol.31, no.=2, pp. 221–239, 1989.

[2] Beal, M.J., Variational Algorithms for Approximate Bayesian Inference, PhD. Thesis, Gatsby Computational Neuroscience Unit, University College London, 2003.