Matrix Inversion Lemma and Information Filter

Mohammad Emtiyaz Khan
Honeywell Technology Solutions Lab, Bangalore, India.

Abstract

In this paper, we discuss two important matrix inversion lemmas and it's application to derive information filter from Kalman filter. Advantages of Information filter over Kalman filter are also discussed.

1 The Model

Consider the following model state-space model:

$$\begin{aligned}\mathbf{x}_{t+1} &= A_t\mathbf{x}_t + B_t\mathbf{w}_t \\ \mathbf{y}_t &= C_t\mathbf{x}_t + \mathbf{v}_t\end{aligned} \tag{1}$$

where, $\mathbf{x}_t \in \Re^n$ is the state vector, $\mathbf{y}_t \in \Re^m$ is the output vector, $\mathbf{w}_t \in \Re^n$ and $\mathbf{v}_t \in \Re^m$ are the state and measurement noise vectors respectively. We assume that the initial state vector and the noise vectors are *i.i.d* Gaussian random variables, $\mathbf{x}_0 \sim N(\mu_0, \Sigma_0)$, $\mathbf{w}_t \sim N(0, Q_t)$, $\mathbf{v}_t \sim N(0, R_t)$, where, Σ_0, Q_t and R_t are symmetric, positive definite matrices. For simplicity, $E\left(\mathbf{v}_t\mathbf{w}_t^T\right) = 0$, $E\left(\mathbf{x}_0\mathbf{w}_t^T\right) = 0$ and $E\left(\mathbf{x}_0\mathbf{v}_t^T\right) = 0$, where $E(\,)$ is the *Expectation* operator. We define, $Y_s \equiv \{\mathbf{y}_1, \ldots, \mathbf{y}_s\}$. Further, we use the following definitions for the conditional expectations of the states and the corresponding error covariances: $\hat{\mathbf{x}}_{t|s} = E\left(\mathbf{x}_t|Y_s\right)$ and $P_{t|s} = E\left((\mathbf{x}_t - \hat{\mathbf{x}}_{t|s})(\mathbf{x}_t - \hat{\mathbf{x}}_{t|s})^T|Y_s\right)$.

2 Matrix-Inversion Lemma

Consider $P \in \Re^{n\times n}$. Assuming the inverses to exist, we have the following Matrix inversion lemmas:

1. $(I + PC^TR^{-1}C)^{-1}P = (P^{-1} + C^TR^{-1}C)^{-1} = P - PC^T(CPC^T + R)^{-1}CP$ (2)
2. $(I + PC^TR^{-1}C)^{-1}PC^TR^{-1} = (P^{-1} + C^TR^{-1}C)^{-1}C^TR^{-1} = PC^T(CPC^T + R)^{-1}$ (3)

The second equation is a variant of Eq. (2). Proof of these are given in Appendix.

3 Classical Kalman filter

The filtered state estimates are given as,

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + P_{t|t-1}C^T(CP_{t|t-1}C^T + R)^{-1}(\mathbf{y}_t - C\hat{\mathbf{x}}_{t|t-1}) \quad (4)$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}C^T(CP_{t|t-1}C^T + R)^{-1}CP_{t|t-1} \quad (5)$$

where the Kalman gain is, $K_t = P_{t|t-1}C^T(CP_{t|t-1}C^T + R)^{-1}$. The one step ahead state prediction is obtained as,

$$\hat{\mathbf{x}}_{t+1|t} = A_t\hat{\mathbf{x}}_{t|t}, \qquad P_{t+1|t} = A_tP_{t|t}A_t^T + B_tQ_tB_t^T \quad (6)$$

with initial condition $\mathbf{x}_{1|0} = 0$ and $P_{1|0} = \Sigma_0$.

4 Information Filter

Define $Z_{t|s} = P_{t|s}^{-1}$ and $\mathbf{z}_{t|s} = P_{t|s}^{-1}\hat{\mathbf{x}}_{t|s}$, called *information matrix* and *information state vector*. As described in (Mutambara 1998), for a Gaussian case, inverse of the covariance matrix (also called Fisher information) provides the measure of information about the state present in the observations.

Direct application of matrix inversion lemma given by Eq. (2) and (3) on the Kalman filter gives us the following correction and prediction equations for information filter:

$$\mathbf{z}_{t|t} = \mathbf{z}_{t|t-1} + C_t^TR_t^{-1}\mathbf{y}_t, \quad Z_{t|t} = Z_{t|t-1} + C_t^TR_t^{-1}C_t \quad (7)$$

$$\mathbf{z}_{t+1|t} = (I - J_tB_t^T)A_t^{-T}\mathbf{z}_{t|t}, \quad Z_{t+1|t} = (I - J_tB_t^T)S_t \quad (8)$$

where $S_t = A_t^{-T}P_{t|t}^{-1}A_t^{-1}$ and $J_t = S_tB_t(B_t^TS_tB_t + Q_t^{-1})^{-1}$. The Kalman gain is given as $K_t = Z_{t|t}C_t^TR_t^{-1}$. For detailed proof see (Anderson and Moore 1979) or (Mutambara 1998).

Another form as given in (Mutambara 1998), is as follows

$$\hat{\mathbf{z}}_{t|t} = \hat{\mathbf{z}}_{t|t-1} + \mathbf{i}_t, \quad Z_{t|t} = Z_{t|t-1} + I_t \quad (9)$$

$$\hat{\mathbf{z}}_{t+1|t} = L_{t+1|t}\hat{\mathbf{z}}_{t+1|t}, \quad Z_{t+1|t} = [A_tZ_{t|t}^{-1}A_t^T + Q_t]^{-1} \quad (10)$$

where $L_{t+1|t} = Z_{t+1|t}A_tZ_{t|t}^{-1}$. Also $\mathbf{i}_t = C_t^TR_t^{-1}\mathbf{y}_t$ and $I_t = C_t^TR_t^{-1}C_t$, and are called "information vector and matrix associated with the observations $\mathbf{y}_t$".

Some advantages of the Information filter over Kalman filter are listed here:

1. Computationally simpler correction step.

2. Although the prediction equations are more complex, prediction depends on a propagation coefficient which is independent of the observations. It is thus easy to decouple and decentralize.

3. There is no gain or innovation covariance matrices and the maximum dimension of a matrix to be inverted is the state dimension, which is usually smaller than the observation dimensions in a multi-sensor system; $(CP_{t|t-1}C^T + R)$ in Kalman filter and $(B_t^T S_t B_t + Q_t^{-1})$ in Information filter. Hence it is always preferable to employ information filter and invert smaller information matrices than use the Kalman filter and invert the larger innovation covariance matrices.

A few more points are discussed in (Anderson and Moore 1979), page 140.

A Proof for matrix inversion lemma

For Eq. 2, first step is immediate. For second step we verify, $(I + PC^T R^{-1}C)[I - PC^T(CPC^T + R)^{-1}C] = I$,

$$\begin{aligned} L.H.S. &= (I + PC^T R^{-1}C)[I - PC^T(CPC^T + R)^{-1}C] \\ &= (I + PC^T R^{-1}C) - (I + PC^T R^{-1}C)[PC^T(CPC^T + R)^{-1}C] \\ &= (I + PC^T R^{-1}C) - (I + PC^T R^{-1}C)(I + C^{-1}RC^{-T}P^{-1})^{-1} \\ &= (I + PC^T R^{-1}C) - PC^T R^{-1}C(C^{-1}RC^{-T}P^{-1} + I)(I + C^{-1}RC^{-T}P^{-1})^{-1} \\ &= I \end{aligned}$$

which completes the proof of Eq. (2).

For Eq. (3), multiply Eq. (2) by $C^T R^{-1}$,

$$(I + PC^T R^{-1}C)^{-1}PC^T R^{-1} = (P^{-1} + C^T R^{-1}C)^{-1}C^T R^{-1} = P - PC^T(CPC^T + R)^{-1}CPC^T R^{-1} \quad (11)$$

Now note that $CPC^T R^{-1} = (CPC^T + R)R^{-1} - I$, substituting which in the above equation, we get Eq. (3).

References

Anderson, D.O. Brian and John B. Moore (1979). *Optimal Filtering*. 1st ed.. Prentice Hall,Inc.. Englewood Cliffs, N.J.

Mutambara, G.O. (1998). *Decentralized Estimation and Control for Multisensor Systems*. 1st ed.. CRC press LLC. 2000 Corporate Blvd., N.W., Boca Raton, Florida 33431.