A Review of Linear Algebra

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Basics

- Column vector $\mathbf{x} \in R^n$, Row vector $\mathbf{x}^T$, Matrix $A \in R^{m\times n}$.
- Matrix Multiplication, $(m \times n)(n \times k) \Rightarrow m \times k$, $AB \neq BA$.
- Transpose A^T, $(AB)^T = B^T A^T$, Symmetric $A = A^T$
- Inverse A^{-1}, doesn't exist always, $(AB)^{-1} = B^{-1}A^{-1}$.
- $\mathbf{x}^T\mathbf{x}$ is a scalar, $\mathbf{x}\mathbf{x}^T$ is a matrix.
- $A\mathbf{x} = \mathbf{b}$, three ways of expressing:
  - $\sum_{j=1}^{n} a_{ij}x_j = b_j, \forall j$
  - $\mathbf{r}_j^T\mathbf{x} = b_j, \forall j$, where $\mathbf{r}_j$ is j^{th} row.
  - $x_1\mathbf{a_1} + x_2\mathbf{a_2} + \ldots + x_n\mathbf{a_n} = \mathbf{b}$ (Linear Combination, l.c.)
- System of equations : Non-singular (unique solution), singular (no solution, infinite solution).

LU factorization

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \Rightarrow A\mathbf{x} = \mathbf{b}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} \Rightarrow U\mathbf{x} = EFG\mathbf{b}$$

$$\underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 & & \\ -2 & 1 & \\ 1 & & 1 \end{bmatrix}}_{G}, \qquad L = G^{-1}F^{-1}E^{-1}$$

LU factorization

- (First non-singular case) If no row exchanges are required, then $A = LU$ (unique).
- Solve $L\mathbf{c} = \mathbf{b}$, then $U\mathbf{x} = \mathbf{c}$
- Another form $A = LDU$.
- (Second non-singular case) There exist a permutation matrix P that reorders the rows, so that $PA = LU$.
- (Singular Case) No such P exist.
- (Cholesky Decomposition) If A is symmetric, and $A = LU$ can be found without any row exchanges, then $A = LL^T$ (also called square root of a matrix). (proof).
- Positive Definite matrix always have a Cholesky decompostion.

Vector Space, Subspace and Matrix

- (Real Vector Space) A set of “vectors" with rules for vector addition and multiplication by real numbers. E.g. $R^1, R^2, \ldots, R^\infty$, Hilbert Space.
- (8 conditions) Includes an identity vector and zero vector, closed under addition and multiplication etc. etc.
- (Subspace) Subset of a vector space, closed under addition and multiplication (should contain zero).
- Subspace “spanned" by a matrix (Outline the concept)

$$x_1 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear Independence, Basis, Dimension

- (Linear Independence, l.i.) If $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$ only happens when $x_1 = x_2 = \ldots = 0$, $\{\mathbf{a}_k\}$ are called linearly independent.
- A set of n vectors in R^m are not l.i. if $n > m$ (proof).
- (Span) If every vector v in V can be expressed as a l.c. of $\{\mathbf{a}_k\}$, then $\{\mathbf{a}_k\}$ are said to span V.
- (Basis) $\{\mathbf{a}_k\}$ are called basis of V if they are l.i. and span V (Too many and unique)
- (Dimension) Number of vectors in any basis is called dimension (and is same for all basis).

Four Fundamental Spaces

Fundamental Theorem of Linear Algebra I

1. $\mathcal{R}(A)$ = Column Space of A; l.c. of columns; dim r.
2. $\mathcal{N}(A)$ = Nullspace of A; All $x : A\mathbf{x} = 0$; dim $n - r$.
3. $\mathcal{R}(A^T)$ = Row space of A; l.c. of rows; dim r.
4. $\mathcal{N}(A^T)$ = Left nullspace of A; All $y : A^T\mathbf{y} = 0$; dim $m - r$.

(Rank) r is called rank of the matrix. Inverse exist iff rank is as large as possible. Question: Rank of $\mathbf{uv}^T$

Orthogonality

- (Norm) $||\mathbf{x}||^2 = \mathbf{x}^T\mathbf{x} = x_1^2 + \ldots + x_n^2$
- (Inner Product) $\mathbf{x}^T\mathbf{y} = x_1y_1 + \ldots + x_ny_n$
- (Orthogonal) $\mathbf{x}^T\mathbf{y} = 0$
- Orthogonal $\Rightarrow$ l.i. (proof).
- (Orthonormal basis) Orthogonal vectors with norm =1
- (Orthogonal Subspaces) $V \perp W$ if $v \perp w, \forall v \in V, w \in W$
- (Orthogonal Complement) The space of all vectors orthogonal to V denoted as $V^{\perp}$.
- The row space is orthogonal to the nullspace (in R^n) and the column space is orthogonal to the left nullspace (in R^m).(proof).

Finally...

Fundamental Theorem of Linear Algebra II

1. $\mathcal{R}(A^T)^\perp = \mathcal{N}(A)$
2. $\mathcal{R}(A)^\perp = \mathcal{N}(A^T)$

Any vector can be expressed as

$$
\begin{aligned}
\text{(1)} \qquad \mathbf{x} &= \underbrace{x_1\mathbf{b}_1 + \ldots + x_r\mathbf{b}_r}_{\mathbf{x}_r} + \underbrace{x_{r+1}\mathbf{b}_{r+1} + \ldots + x_n\mathbf{b}_n}_{\mathbf{x}_n} \\
\text{(2)} \qquad &= \mathbf{x}_r + \mathbf{x}_n
\end{aligned}
$$

Every matrix transforms its row space to its column space

(Comments about pseudo-inverse and invertibility)

Gram-Schmidt Orthogonalization

- (Projection) of $\mathbf{b}$ on $\mathbf{a}$ is $\frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}$, for unit vector $(\mathbf{a}^T\mathbf{b})\mathbf{a}$
- (Schwartz Inequality) $|\mathbf{a}^T\mathbf{b}| \leq ||\mathbf{a}||||\mathbf{b}||$
- (Orthogonal Matrix)$Q = [\mathbf{q}_1 \ldots \mathbf{q}_n]$, $Q^T Q = I$. (proof).
- (Length preservation) $||Q\mathbf{x}|| = ||x||$ (proof).

Given vectors $\{\mathbf{a}_k\}$, construct orthogonal vectors$\{\mathbf{q}_k\}$

1. $\mathbf{q}_1 = \mathbf{a}_1/||\mathbf{a}_1||$
2. for each j, $\mathbf{a}_j^{'} = \mathbf{a}_j - (\mathbf{q}_1^T\mathbf{a}_j)\mathbf{q}_1 - \ldots - (\mathbf{q}_{j-1}^T\mathbf{a}_j)\mathbf{q}_{j-1}$
3. $\mathbf{q}_j = \mathbf{a}_j^{'}/||\mathbf{a}_j^{'}||$

QR Decomposition (Example)

Eigenvalues and Eigenvectors

- (Invariance)$A\mathbf{x} = \lambda\mathbf{x}$.
- (Characteristics Equation) $(A - \lambda I)\mathbf{x} = 0$ (Nullspace)
- $\lambda_1 + \ldots + \lambda_n = a_{11} + \ldots + a_{nn}$.
- $\lambda_1 \ldots \lambda_n = det(A)$.
- ($A = S\Lambda S^{-1}$) Suppose there exist n linear independent eigenvectors for A. If S is the matrix whose columns are those independent vectors, then $A = S\Lambda S^{-1}$ where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$.
- Diagonalizability is concerned with eigenvectors, and invertibility is concerned with eigenvalues.
- (Real symmetric matrix) Eigenvectors are orthogonal. So $A = Q\Lambda Q^T$. (Spectral Theorem)

Singular Value Decomposition

Any matrix can be factorized as $A = U\Sigma V^T$. Insightful? Finish.

Finish

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