PROBABILITY SPACE

AN IMPORTANT AND USELESS CONCEPT

COIN FLIP:

Sample space: $\Omega = \{H, T\}$

Event space:

$$\mathcal{F} = \{\{H,T\}, \{H\}, \{T\}, \phi\}$$

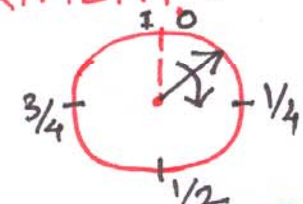
Probability Measure:

$$P_r^1 : \mathcal{F} \to [0,1]$$

$$\left[P_r^1(\{H\}) \triangleq \boxed{p},\ P_r^1(\{T\}) \triangleq \boxed{q},\ P_r^1(\{H,T\}) \triangleq 1,\ P_r^1(\varphi) \triangleq 0 \right]$$

TYCHE'S EXPERIMENT:

$$\Omega = (0,1]$$



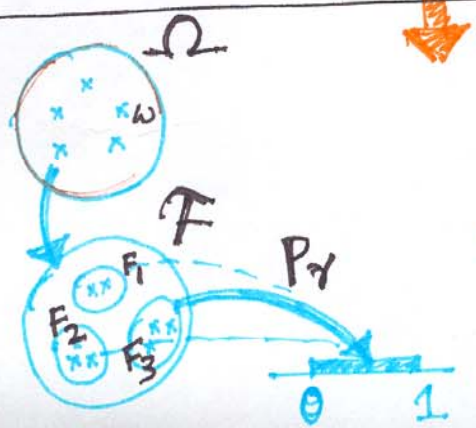
$$\mathcal{F} = \{\text{all } (a,b)\},\quad a,b \in \Omega$$

(Borel)

$$P_r^2 : \mathcal{F} \to [0,1]$$

$$\left[P_r^2(\{(a,b)\}) \triangleq \int_a^b \boxed{f(\gamma)}\, d\gamma \right.$$

$$\left. \text{where } f(\gamma) = 1,\ \forall \gamma \in [0,1] \right]$$



Axioms (EVENT SPACE)	(Probability Measure)
① if $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$	① $P_r(F) \geq 0$
② $F_1 \in \mathcal{F},\ F_2 \in \mathcal{F} \Rightarrow F_1 \cup F_2 \in \mathcal{F}$	② $P_r(\Omega) = 1$
③ $\bigcup_{i=1}^{n} F_i \in \mathcal{F}$	③ $P_r\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{n} P(F_i)$ if $\bigcap_{i=1}^{n} F_i = \varphi$

INDEPENT EVENTS & CONDITIONAL PR.

$F, G \in \mathcal{F}$ are independent events: $\underbrace{\Pr(\boxed{F \cap G})}_{\cancel{\Pr(F,G)}} = \Pr(F)\Pr(G)$

$\Pr\left(\bigcap_{i=1}^{n} F_i\right) = \prod_{i=1}^{n} \Pr(F_i)$

Different from mutually-exclusive

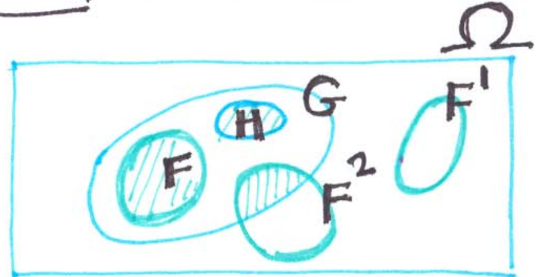
$\boxed{F \cap G = \emptyset} \Rightarrow \Pr(F \cup G) = \Pr(F) + \Pr(G)$

Conditional Probability
(A different measure)

$\Pr(F/G)$

"Probability that event F occurs given that the event G occured"

$$\Pr(F/G) = \frac{\Pr(\boxed{F \cap G})}{P(G)}$$



★ $\Pr(F^1/G) \boxed{\triangleq} 0 \rightarrow \Pr(G^c/G) = 0$

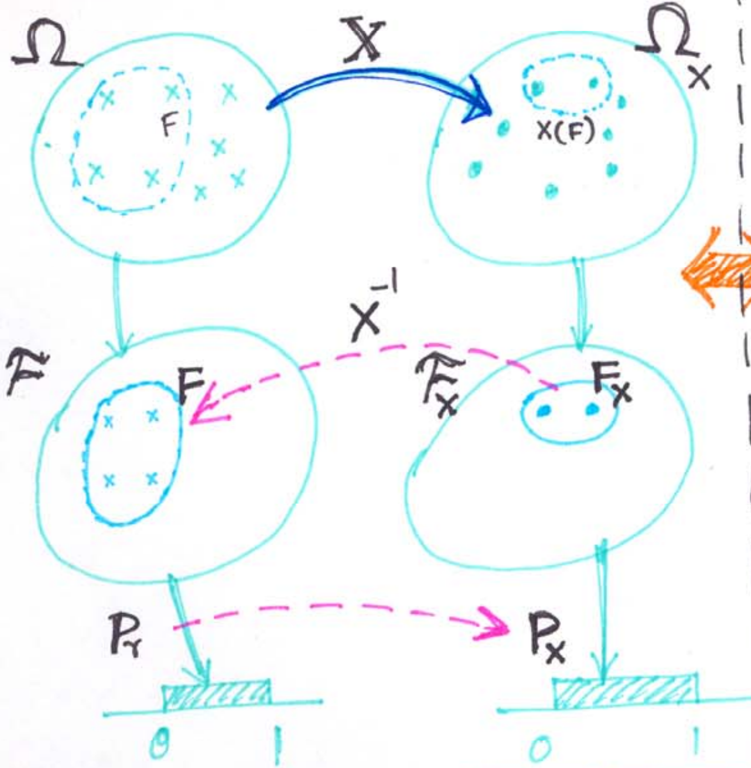
$\Pr(F/G) = \Pr(F \cap (G \cup G^c)/G)$

$= \Pr(F \cap G/G) + \underbrace{\Pr(F \cap G^c/G)}_{0}$

★ $\dfrac{\Pr(F \cap G)}{\Pr(\Omega \cap G)} \boxed{\triangleq} \dfrac{\Pr(F \cap G/G)}{\Pr(\Omega \cap G/G)}$

$\dfrac{\Pr(F \cap G)}{\Pr(G)} = \Pr(F \cap G/G)$

RANDOM VARIABLES & its probability DISTRIBUTION



· A random variable is a function $X : \Omega \to \Omega_X$, such that $X^{-1}(\mathcal{F}_X) \in \mathcal{F}$.
i.e. every event is "preserved."

DERIVED DISTRIBUTION

$$P_X(F_X) \triangleq Pr\left(X^{-1}(F_X)\right) = Pr\left(\boxed{\{\omega : X(\omega) \in F_X\}}\right)$$

Discrete r.v.	Continuous r.v.
$p_X(x_k) = P_X\left(\underbrace{\boxed{\{X = x_k\}}}_{\{\omega : X(\omega) = x\}}\right)$	$f_X(x_k) = \lim_{\delta \to 0} (\cdot)$ $P_X\left(\{x \in (x+\delta, x-\delta)\}\right)$
Probility Mass f^n	Probability density f^n
(1) $\sum_{x \in \Omega_X} p_X(x) = \boxed{1}$	(1) $\int_{\Omega_X} f_X(x) = \boxed{1}$
(2) $p_X(x) \geq 0$	(2) $f_X(x) \geq 0$

Cumulative Distribution Function

$$F_X(\alpha) = P_X(\{x : x \leq \alpha\})$$

≤ 1

EXAMPLE

Discrete

BINARY

$\Omega_X = \{0, 1\}$,

$p(0) = 1 \; , \; p(1) = 1 - p$

UNIFORM

$\Omega_X = \{0, 1, \cdots, n-1\}$

$p(k) = 1/n \; , \; k \in \Omega_X$

BINOMIAL:

$\Omega_X = \{0, 1, 2\}$, $X = Y + Z$

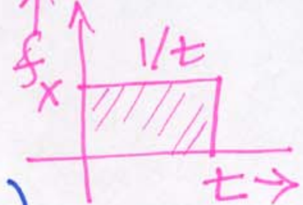
Binary

$p(k) = {}^2C_k \, p^k (1-p)^k$

${}^2C_1 \cdot p(1-p) \leftarrow 1 = \begin{cases} 0+1 \\ 1+0 \end{cases}$

Continuous

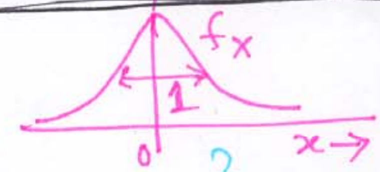
UNIFORM



$\Omega_X = (0, t)$

$f_X(x) = \begin{cases} 1/t \, , & 0 \le x \le t \\ 0 \, , & \text{otherwise} \end{cases}$

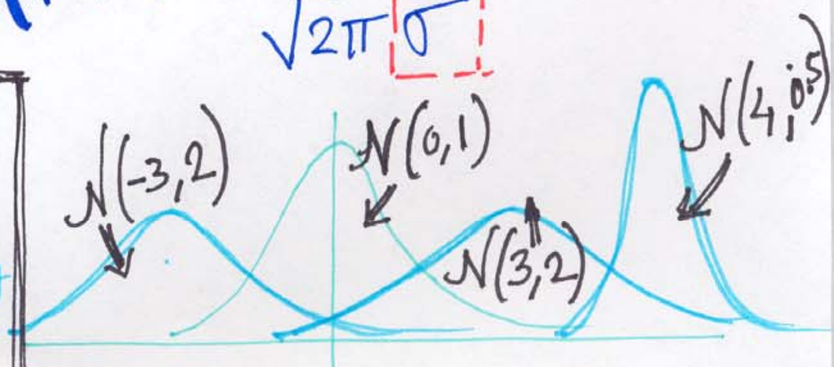
Gaussian



$\Omega_X = \mathbb{R}$

$$f_X(x) = \boxed{\frac{1}{\sqrt{2\pi}}} \, e^{-\frac{x^2}{2}}$$

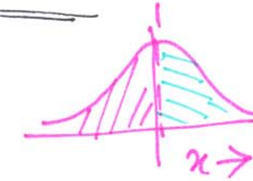
$$\mathcal{N}(m, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\boxed{\sigma}} \, e^{-\frac{(x \boxed{-m})^2}{2\boxed{\sigma^2}}}$$



EXPECTATION IS GOOD....

$$\underset{\boxed{X}}{E}\left[\boxed{g}(X)\right] = \begin{cases} \displaystyle\sum_{x \in \Omega_X} g(x) \cdot \boxed{p_X}(x) \\ \displaystyle\int_{\Omega_X} g(x) \boxed{f_X(x)} dx \end{cases}$$

Mean (m)



$$\underset{X}{E}(X) = \begin{cases} \sum x \, p_X(x) \\ \int x \, f_X(x) dx \end{cases}$$

Variance (σ^2)

$$\underset{X}{E}\left[(X-m)^2\right] = \int_{\Omega_X} (x-m)^2 f_X(x) dx$$

$$F_X(x) \text{ "}\alpha\text{" } M_X(ju) = \underbrace{1 + ju \underset{X}{E}(X) + \frac{ju}{2} \underset{X}{E}(X^2)}_{\text{Gaussian.}} + \cdots \quad \vdots$$

Higher "moments"

WEAK	STRONG
1. $\Pr(X \geq 0) = 1 \Rightarrow EX \geq 0$	$\Pr(F) \geq 0 \quad \forall F$
2. $\Pr(X = c) = 1 \Rightarrow EX = C$	$\Pr(\Omega) = 1$
3. $E(aX + bY) = aEX + bEY$	$P(A \uplus B) = P(A) + P(B)$
4. $E\left(\lim_{n\to\infty} X_n\right) = \lim_{n\to\infty} EX_n$	$P\left(\lim_{n\to\infty} F_n\right) = \lim_{n\to\infty} P(F_n)$
5. $E(XY) = E(X) \cdot E(Y)$ (uncorrelated)	$P(A \cap B) = P(A) \cdot P(B)$

TWO RESULTS

A WEAK LAW OF LARGE NUMBER

$X_0 \quad X_1 \quad X_2 \quad X_3 \quad \cdots\cdots$

Uncorrelated

Each m, σ^2

$$\bar{X}_K = \frac{1}{\boxed{n}} \sum_{n=1}^{\boxed{K}} X_n$$

As $K \to \infty$

$$\boxed{\begin{array}{l} E(\bar{X}_K) \to m \\ \sigma^2_{\bar{X}_K} \to 0 \end{array}}$$

{ statistics ! }

A (STRONG) CENTRAL LIMIT THEOREM

$X_0 \quad X_1 \quad X_2 \quad X_3 \quad \cdots\cdots$

Independent

Each m, σ^2

$$\tilde{X}_K = \frac{1}{\boxed{\sqrt{n}}} \sum_{n=1}^{\boxed{K}} (X_n - m)$$

As $K \to \infty$

$$\cancel{E(\bar{X}_K) \to m}$$

$$\boxed{F_{\bar{X}_K} \to \mathcal{N}(m, \sigma^2).}$$

{ Thermal Noise
Brownian Motion