Sampling methods for Probit Regression and Application to GMRF

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Overview

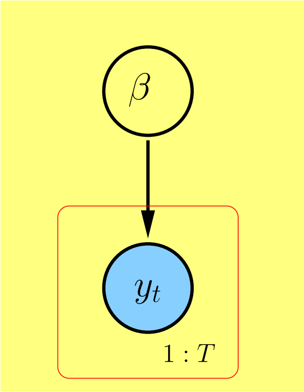
- Probit Model
- Sampling methods : A&C (1993), H&H (2006)
- Comparison
- Modifications for GMRF

Probit Model

For $t = 1, \ldots, T$, $y_t \in \{0, 1\}$, $\beta \in \mathcal{R}^d$,

(1) $$Pr(y_t = 1|\beta) \;=\; \Phi(x_t\beta) \quad \text{where } \beta \sim \pi(\beta)$$

where $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \exp(-u^2|2)du$.

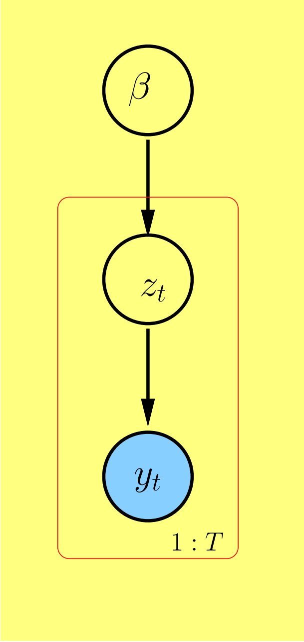


Using Auxiliary variables, we can re-write:

(2) $$y_t = \begin{cases} 1 & \text{if } z_t > 0 \\ 0 & \text{otherwise} \end{cases}$$

(3) $$z_t = x_t\beta + \epsilon_t, \text{ where } \epsilon_t \sim \mathcal{N}(0, 1)$$

(4) $$\beta \sim p(\beta)$$



Gibbs Sampling I - Albert and Chib (1993)

If we assume prior $\beta \sim \mathcal{N}(b, v)$ and $X = [x_i' x_2' \ldots x_n']'$

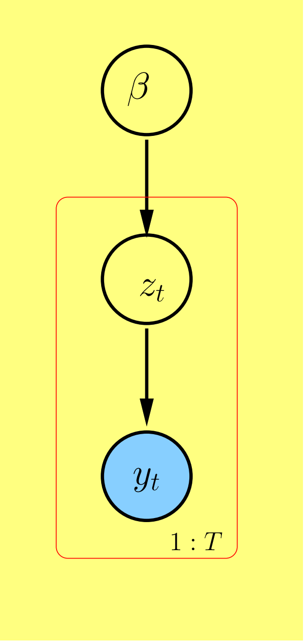
(5) $$p(\beta|z_{1:T}) \propto p(\beta) \prod_{t=1}^{T} p(z_t|\beta) = \mathcal{N}(\beta; B, V)$$

$$B = V(v^{-1}b + X'z_{1:T})$$

$$V = (v^{-1} + X'X)^{-1}$$

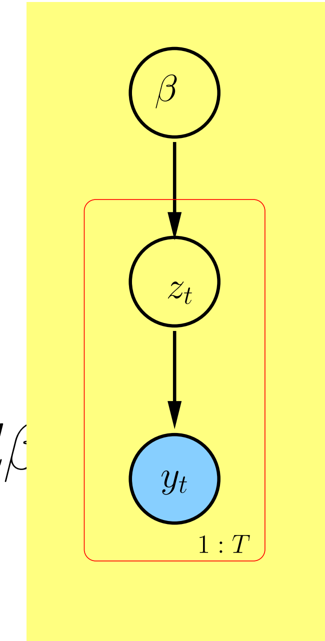
(6) $$p(z_{1:T}|\beta, y_{1:T}) \propto \prod_{t=1}^{T} p(y_t|z_t)p(z_t|\beta)$$

$$= \begin{cases} \mathcal{N}(x_t\beta, 1)I(z_t \geq 0) & \text{if } y_t = 1 \\ \mathcal{N}(x_t\beta, 1)I(z_t < 0) & \text{otherwise} \end{cases}$$



Gibbs Sampling II - Holmes and Held (2006)

- An alternative - Update joint simultaneously:



$$
\begin{aligned}
p(\beta, z_{1:T}|y_{1:T}) &= p(z_{1:T}|y_{1:T})p(\beta|z_{1:T}) \\
p(z_{1:T}|y_{1:T}) &\propto p(y_{1:T}|z_{1:T}) \int_{\beta} p(z_{1:T}|\beta)p(\beta)d\beta \\
&= p(y_{1:T}|z_{1:T})f(z_{1:T}) \int_{\beta} p(\beta|z_{1:T})d\beta \\
p(z_{1:T}|y_{1:T}) &= I(z_{1:T}, y_{1:T})\mathcal{N}(0, I_T + XVX')
\end{aligned}
$$

- Sampling from a multivariate truncated Gaussian!
- However this can be simplified as described in HH(2006).

Our Work

- Implement A&C and H&H Gibbs sampler and compare.
- Modifications for GMRF.
- Modifications for polychotomous variables.
- Application to CGH microarray data.

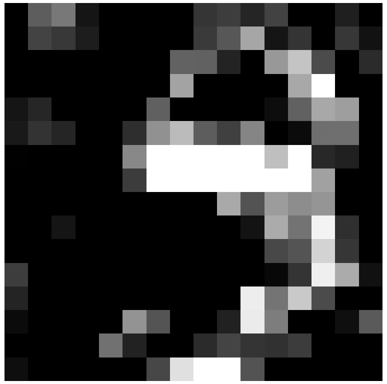
Comparison: A&C and H&H

Mean of β

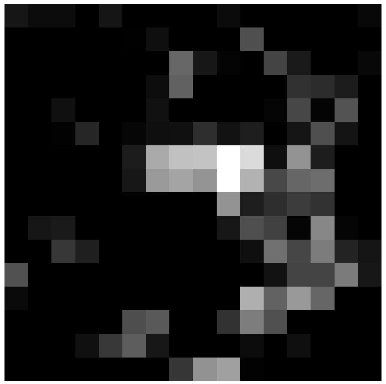
(189 Images (16x16), 300 iterations, $\beta \sim \mathcal{N}(0, I)$)

Template Images

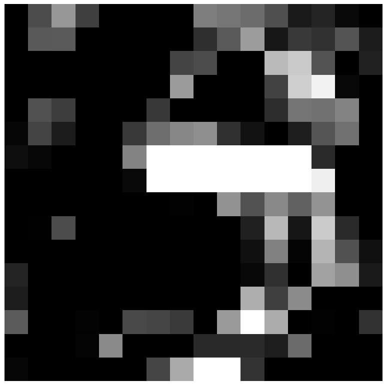
HH − MS code



AC − MTs code



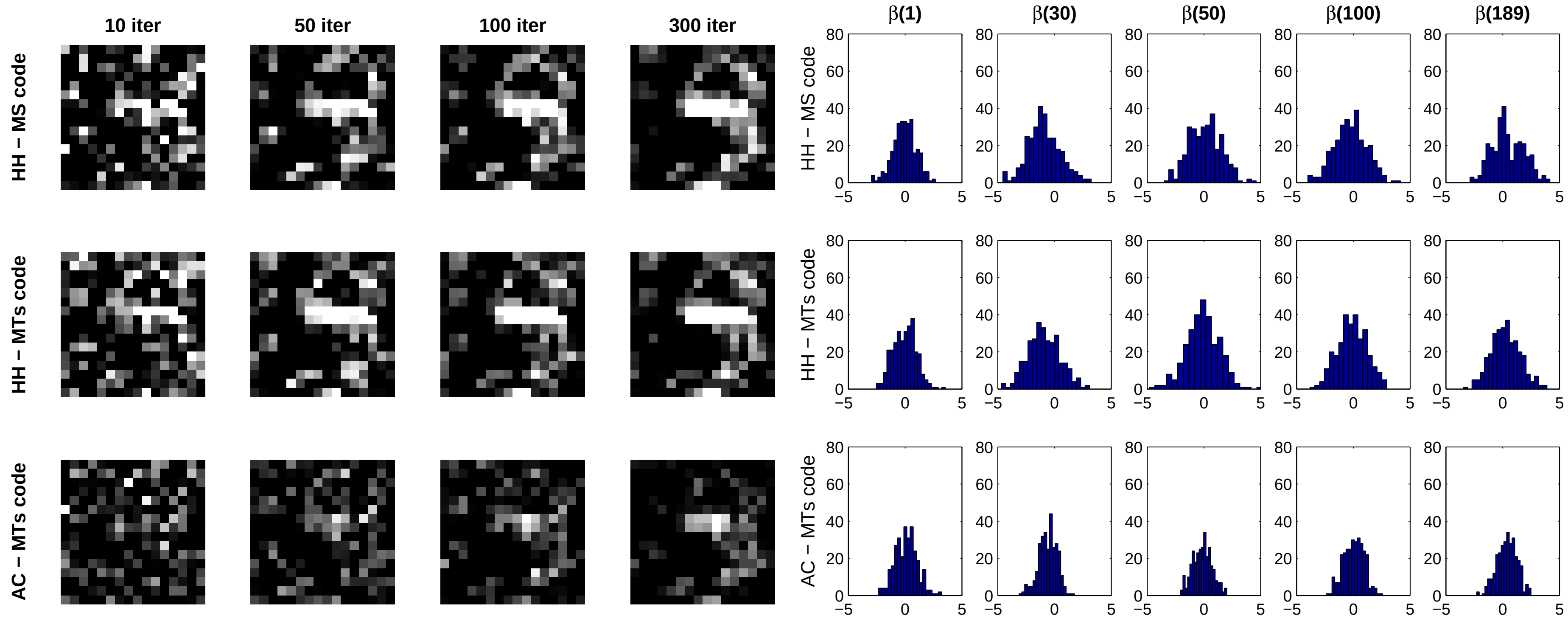
HH − MTs code



Comparison: A&C and H&H

Convergence

Distribution of β



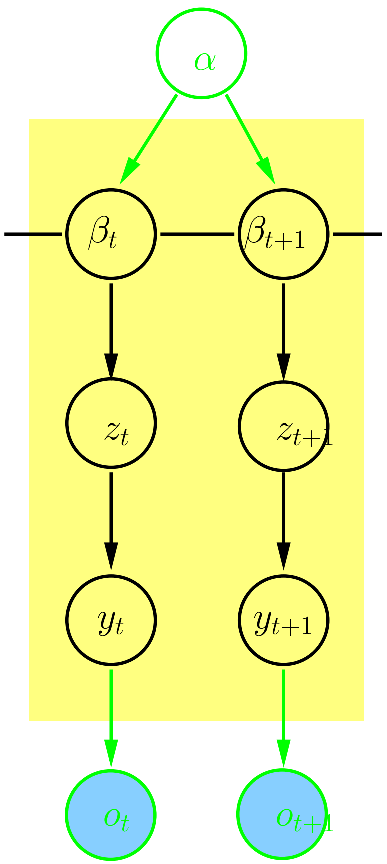
Modification for GMRF

We extend these results to the case when β follows a Gaussian Markov Random Field (suggested by Kevin Murphy):

$$y_t = \begin{cases} 1 & \text{if } z_t > 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

$$z_t = \beta_t + \epsilon_t \quad \text{where } \epsilon_t \sim \mathcal{N}(0,1) \tag{8}$$

$$\beta_{1:T} \sim \exp\left[-\frac{1}{2} \sum_{t=1}^{T-1} \frac{(\beta_t - \beta_{t-1})^2}{\alpha} \right] \tag{9}$$



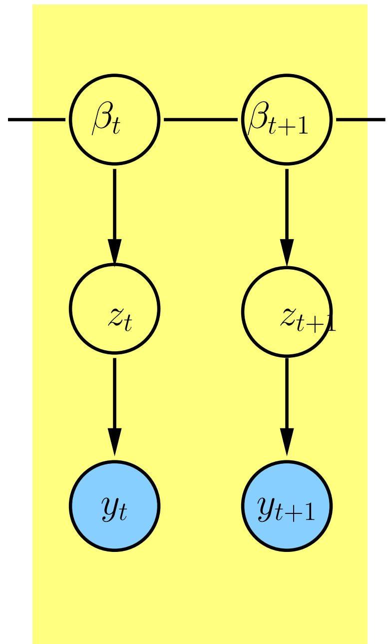
Distribution of $\beta = vec(\beta_{1:T})$

$$\beta_{1:T} \propto \exp\left[-\frac{1}{2} \beta' W \beta \right] \tag{10}$$

Modification for GMRF : Following A&C

If we assume prior $\beta_{1:T} \sim \mathcal{N}(0, v)$:

$$\text{(11)} \qquad p(z_t|\beta_t, y_t) \propto \begin{cases} \mathcal{N}(x_t\beta_t, 1)I(z_t \geq 0) & \text{if } y_t = 1 \\ \mathcal{N}(x_t\beta_t, 1)I(z_t < 0) & \text{otherwise} \end{cases}$$



$$\begin{aligned} \text{(12)}\ p(\beta_{1:T}|z_{1:T}) &\propto p(\beta_0)\prod_{t=1}^{T} p(z_t|\beta_t)p(\beta_t|\beta_{t-1}) \\ &= \mathcal{N}(\beta_{1:T}; B, V)\mathcal{N}(\beta_{1:T}; 0, W) \\ &= \mathcal{N}(\beta_{1:T}; B, V + W) \end{aligned}$$

Modification for GMRF : Following H&H

$$
\begin{aligned}
p(\beta_{1:T}, z_{1:T}|y_{1:T}) &= p(z_{1:T}|y_{1:T})p(\beta_{1:T}|z_{1:T}) \\
p(z_{1:T}|y_{1:T}) &\propto p(y_{1:T}|z_{1:T}) \int_{\beta_{1:T}} p(z_{1:T}|\beta_{1:T})p(\beta_{1:T})d\beta_{1:T} \\
&= p(y_{1:T}|z_{1:T})f(z_{1:T}) \int_{\beta_{1:T}} p(\beta_{1:T}|z_{1:T})d\beta \\
p(z_{1:T}|y_{1:T}) &= I(z_{1:T}, y_{1:T})\mathcal{N}(0, I_T + XVX')
\end{aligned}
$$

Summary

We presented

- A comparison two sampling methods for Probit models
- Suggested modifications for GMRF.

We wish to finish

- Modification for polychotomous variable model
- Application to CGH microarray data.

Acknowledgments: Thanks to Kevin Murphy, Mark Schmidt, and (of course) Arnaud.