Consistency Check for Linear Gaussian Filters

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Our objective is to check the consistency[1] of the linear Gaussian filters. Suppose we have an unbiased estimate $\hat{\mathbf{x}}_t$ of $\mathbf{x}_t$ with covariance P_t, obtained using a linear Gaussian filter. Then the error $\mathbf{e}_t = \mathbf{x}_t - \hat{\mathbf{x}}_t$ will be distributed as $\mathcal{N}(0, P_t)$. For such cases (unbiased estimate), the consistency means that the error statistically corresponds to it's covariance matrix P_t, the filter is said to be consistent(in other words, sample variance approaches the P_t as number of samples goes to infinity).

As discussed in (Bar-Shalom and Fortmann 1988), one measure of consistency is the normalized state error squared variable, defined as,

$$\varepsilon_t = \mathbf{e}_t^T P_t^{-1} \mathbf{e}_t \tag{2}$$

ε_t is $\chi^2(n)$, where n is equal to number of states, i.e. $\dim(\mathbf{x}_t)$. In addition, $E(\varepsilon_t) = \dim(\mathbf{x}_t)$, which can be easily proved:

$$\begin{aligned} E(\varepsilon_t) &= E(\mathbf{e}_t^T P_t^{-1} \mathbf{e}_t) = E[\text{trace}(P_t^{-1} \mathbf{e}_t \mathbf{e}_t^T)] = \text{trace}[P_t^{-1} E(\mathbf{e}_t \mathbf{e}_t^T)] \\ &= \text{trace}[P_t^{-1} P_t] = \dim(\mathbf{x}_t) \end{aligned} \tag{3}$$

Instead of checking the consistency for $\mathbf{e}_t$, we can test whether ε_t follows a Chi-square distribution or not, which will be comparatively easier as it is a scalar. This test can be done by using the confidence interval, as described in (Blackman 1986). The confidence intervals are defined by placing upper and lower limits so that:

$$\Pr\left[f_L(\alpha, \bar{\varepsilon}_t) \le \varepsilon_t \le f_U(\alpha, \bar{\varepsilon}_t)\right] = 1 - \alpha \tag{4}$$

where $\bar{\varepsilon}_t$ is an estimate, obtained statistically with Monte Carlo simulations and α is the allowable probability of error (note!). Based upon simulation a claim could be made that the value lies within the interval with probability $1-\alpha$. This relationship is either true or not, and performing many Monte Carlo simulations will tell us whether the quantity falls withing the interval with a probability of $1-\alpha$ or not.

The test can be summarized as follows:

1. Choose a confidence interval α (for e.g. 0.05). Note that it's the probability of error.

[1] The estimate is said to be consistent (Kay 1993), if for any $\varepsilon > 0$, (Here N is the number of Monte-Carlo simulations)

$$\lim_{N\to\infty} \Pr\left((\hat{\mathbf{x}}_t - \mathbf{x}_t) > \varepsilon\right) = 0 \tag{1}$$

2. Perform a Monte Carlo simulation for N (greater than 50), and compute the mean

$$\bar{\varepsilon}_t = \frac{1}{N}\Sigma_{i=1}^{N}\varepsilon_t^i \tag{5}$$

3. Find the limits for a Chi-square distribution as follows ($n = (dim)\mathbf{x}_t$),

$$\frac{N\bar{\varepsilon}_t}{\chi^2(n,\alpha/2)} \leq \varepsilon_t \leq \frac{N\bar{\varepsilon}_t}{\chi^2(n,1-\alpha/2)} \tag{6}$$

4. Check whether ε_t within the interval for all time points.

References

Bar-Shalom, Y. and Thomas E. Fortmann (1988). *Tracking and Data Association*. Academix Press, Inc.. San Diego, CA, USA.

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Kay, S. M. (1993). *Statistical Signal Processing*. Prentice Hall. New-York.